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MATHEMATICS

Grade 12

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Author's Profile

Mathematics is a necessary avenue to scientific knowledge which opens new vistas of mental activity. A sound knowledge of Mathematics is must for a modern scientist, information technologist, financial specialist and/or an engineer to attain new dimensions in all aspects of professional practices. Applied mathematics is alive and very vigorous. That ought to be reflected in our teaching. In our own teaching we became convinced that the textbook is crucial. It must provide a framework into which the applications will fit in. A good course has a clear purpose, and you can sense that it is there. It is a pleasure to teach a subject when it is moving forward, and this one is-but the book has to share in that spirit and help to establish it.

This book is a self-contained comprehensive volume covering the entire ambit of the course of Mathematics XII offered by Khyber Pakhtunkhwa Textbook Board, Peshawar. Prof. Mumtaz Khan enjoys a rich and diversified experience of 36 years of curriculum development in the specialized areas of Pure Mathematics, Applied Mathematics, Statistics, Quantitative Research and Operational Research, preparing and delivering lectures at undergraduate, graduate and post-graduate level in Engineering, computer science, business administration and information technology, supervising research at undergraduate, graduate and post-graduate level. He is author of many books being taught at different universities and colleges at graduate and undergraduate level. He remained associated with the University Of Engineering & Technology Peshawar since 1976.

One of his areas of expertise is the application of mathematical and statistical techniques to engineering, information technology, computer science, business administration, economics and medicine.

This book is written in a lucid, easy to understand language. Each topic has been thoroughly covered in scope, content and also from the examination point of view. For each topic several worked out examples are carefully selected and presented to cover all aspects of the topic. This is followed by practice exercise with answers to the problems and hints to the difficult ones.

This book provides the computer software "Introduction to Symbolic Package: MAPLE" which remains optional for the time being. The draft of the package, he writes, is self-explanatory. If the students want to involve this package in their studies, they will feel pleasure in solving exercises of the units, such as matrices, system of linear equations, derivatives, integration, line equations, parabola, ellipse, hyperbola, differential equations, partial derivatives, approximate solutions and their graphical views.

- ***Students may write him at: profmumtaz@hotmail.com for any query about any topic discussed in this book.***

بِسۡمِ اللّٰہِ الرَّحۡمٰنِ الرَّحِیۡمِ ۝

ترجمہ: شروع اللہ کے نام سے جو بڑا مہربان، نہایت رحم والا ہے۔

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Introduction To Symbolic Package: MAPLE

This unit tells us, how to

- recognize the MAPLE environment and basic MAPLE commands.
- use online MAPLE help.
- use MAPL Ecommands to simplify polynomials, such as factoring polynomial, expanding an expression, simplifying an expression, simplifying a rational expression and substituting in an expression.
- use MAPLE commands to view two-dimensional graph, domain and range and parametric equations.
- use MAPLE commands to simplify a matrix, vector entry arrangement, matrix operations, inverse and transpose of a matrix.
- use MAPLE as a calculator.

1.1 Introduction

In this introductory course, you will become familiar with and comfortable in the Maple environment. You will learn how to use **"context menus"** and **"palettes"** to perform powerful analysis and create high-impact graphics with only a minimal knowledge of commands. This course will give you the tools, you need to get started quickly, and a solid foundation upon which to build your future Maple explorations.

1.1.1 *Maple Environment*

MAPLE is a powerful mathematical software package. It can be used to obtain both symbolic and numerical solutions of problems in arithmetic, algebra, and calculus and to generate plots of the solutions it generates.

If you want to be able to use MAPLE to solve mathematical problems, start the program with commands and then carry out each step given in the subsequent sections.

Open Maple : If you are on EOS (a UNIX-based operating system at North Carolina State University), then:

- Log on to your EOS account.
- Bring up the application menu (point to an open space on the desktop and use the middle mouse button) and choose "Mathematics < Maple-(14)".

You should eventually see a large window headed by " Maple 14 or 11 or any " with a smaller window headed by Untitled (1)- [Server 1]." The smaller window is the worksheet. The MAPLE command prompt

[>

will show at the upper left of the worksheet by clicking "prompt symbol" on a toolsbar.

When you are finished with the MAPLE session, you will leave the program by selecting "Exit" under the "File " menu (upper left of the MAPLE toolsbar).

Saving Your Program

As with any other program application (word processing, spreadsheeting, etc), you should get in the habit of saving your work frequently, so if the system crashes you can recover most of what you did.

- Choose "Save as" under the "File" menu. Make "Tutorial1.mws" the name of your file.

This name should then appear at the top of your worksheet. As you work, frequently choose "Save" under the "File" menu or type CONTROL-S (hold down the CTRL key and type S) to save your recent work.

Getting Help

Go to the "Help" menu (it's at the top right of the MAPLE toolsbar) and select "Contents". You will see a number of topics in black lettering (What's New, "Mathematics," etc.) and " introduction " and "Worksheet Interface" in green lettering. You can click on any title with green letters to open that section of the Help file. For example,

- Click on "Introduction" and read through the examples to get your first look at what MAPLE does.
- When you are finished, click on the upper left corner of the "Intro" window and select "Close" to get rid of the window.

Printing

If you want to print your program, then:

- Select"Print" under the "File" menu, note the name of the file that will be created when you execute the print command.

- Go into your terminal window and issue the command to print the file required.

1.1.2 MAPLE Commands

A MAPLE command is a statement of a calculation followed by a semicolon (the result is displayed on the screen) or a colon (the result is stored but not displayed). Following are some commonds followed by the displayed results. Enter the commands on your worksheet and verify the given results. When you get to "Save the file," select "Save" under the "File" menu or type CTRL-S. For example,

> $14+15;$

$$29 \tag{1}$$

> $8^2+5\cdot 6+2;$

$$96 \tag{2}$$

> $3\cdot(6+9);$

$$45 \tag{3}$$

CTRL-S

If you don't include a semicolon or colon at the end of a command, MAPLE will interpret the next command line as a continuation of the previous command.

Arithmetic Operations : The symbols +,-,*,/, and ^ (or **) denote addition, subtraction, multiplication, division, and exponential (4^2=4**2=4^2=16). When a string of operations are specified in a command, MAPLE first does exponentiations, then multiplications and divisions, then additions and subtractions. To change the order, use parentheses.

Editing Commands : If you make a mistake in a command (like forgetting a semicolon) or want to change a command, you can go back and edit the command with the cursor and mouse as you would a word-processor text.

Exact Arithmetic and Floating-Point Arithmetic: MAPLE calculates-fractions (exact arithmetic) unless you specify that you want decimals (floating-point arithmetic) with the evalf function ("evalf" stands for "evaluate using floating-point arithmetic").

> $\frac{25}{27} + \frac{3}{51};$

$$\frac{452}{459} \tag{4}$$

> $evalf\left(\frac{25}{27} + \frac{3}{51}, 3\right);$

$$0.985 \tag{5}$$

CTRL-S

The argument 3 in the evalf command specifies the number of significant figures you want in the result. If you omit this command, you will get ten significant figures:

> $evalf\left(\frac{25}{27} + \frac{3}{51}\right);$

$$0.9847494553 \tag{6}$$

1.1.3 On-Line Help

You can get help with MAPLE syntax by using the HELP menu, as described previously. If you have a question about a particular command, you can quickly get help by typing a question mark followed by the name (no semicolon). For example,

> *?differentiation:*

will open a window containing information about what the "differentiation" f function does and how to use it. Click on the little "Cross" box at the upper left of the window to close down the little window.

Enlistment of Variables: Use the colon-equal symbol (:=) to define variables-that is, to assign values to them. Once you have defined a variable, simply typing its name will show its value, and using the name in a formula will cause the value to be substituted. For example,

> $A := 25; B := 145;$

$$A := 25$$
$$B := 145 \quad (7)$$

If you want to string commands together on the same line, then:

> $A := 5; B := 10; C := 12;$

$$A := 5$$
$$B := 10$$
$$C := 12 \quad (8)$$

> $2 \cdot A + 4 \cdot B + 5 \cdot C;$

$$110 \quad (9)$$

> $\frac{4 \cdot A}{B};$

$$2 \quad (10)$$

CTRL-S

Number of Digits (significant figures) in Your Results: When you do floating point arithmetic, MAPLE defaults to 10 significant figures:

> $evalf\left(\frac{5}{2}\right);$

$$2.500000000 \quad (11)$$

You can change the default by setting a variable "Digits" equal to the desired number of significant figures:

> $Digits := 4;$

$$Digits := 4 \quad (12)$$

> $evalf\left(\frac{5}{2}\right);$

$$2.500 \quad (13)$$

Built-in-Functions: MAPLE has many built-in-mathematical functions including sin, cos, tan, exp, ln, and log10. If you want to calculate the value of one of these functions, then you must use the evalf command. For example,

> ln(10);

$$\ln(10) \quad (14)$$

> evalf(ln(10), 4);

$$2.303 \quad (15)$$

Expressions and Functions: An expression is a string of constants, variables, and mathematical operators(+ , -, *, /, ^, =, ln, sin, ...). The following are the expressions:

$\frac{1}{2}$

$$\frac{1}{2} \tag{16}$$

$4 \cdot x^2 - 5 \cdot \ln(x)$

$$4x^2 - 5\ln(x) \tag{17}$$

$5 \cdot x^2 - 12 \cdot y^2 = \cos(x \cdot y)$

$$5x^2 - 12y^2 = \cos(xy) \tag{18}$$

A function is a relationship for a variable (the dependent variable /output in terms of one or more other variables (the independents variables/inputs). The following are the the functions:

$y(x) = 3 \cdot x + 2$

$$y(x) = 3x + 2 \tag{19}$$

$z(x, y) = \frac{2 \cdot x}{y}$

$$z(x, y) = \frac{2x}{y} \tag{20}$$

MAPLE handles expressions and functions in completely different ways, which can lead to a great deal of confusion, when relations like this are encountered:

$y = 6 \cdot x^2 - 15 \cdot x + 4$

$$y = 6x^2 - 15x + 4 \tag{21}$$

This could be an expression relating x and y or the definition of y as a function of x. To do anything to or with this relation (like substituting a value of x into it, or solving it for one variable in terms of the other one, you must know whether the relation is an expression or a function.

In the next two sub sections, we show how expressions and functions are defined and evaluated.

Defing Expressions and Substituting Values into Them: Use the colon-equal (:=) to define an expression, and the "subs " function to substitute a value into it. For example,

> $f := x^2;$

$$f := x^2 \tag{22}$$

> $f;$

$$x^2 \tag{23}$$

> $subs(x = 5, f);$

$$25 \tag{24}$$

Typing [> f(5); will not work for the latter calculation. If you want to use functional notation like that, you need to use the next method to define a function.

Defining Functions and Substituting argument values into Them: Use the colon-equal (:=) and the arrow

symbol -> to define a function, and functional notation to substitute a value into it. For example,

> $g := x \to \frac{1}{(x+1)}$;

$$g := x \to \frac{1}{x+1} \quad (25)$$

> $g(x)$;

$$\frac{1}{x+1} \quad (26)$$

> $g(4)$;

$$\frac{1}{5} \quad (27)$$

CTRL-S

Converting Expressions into Functions : It is noted that previously defined expression $f = x^2$ cannot be treated as a function.

> $f(5)$;

$$x(5)^2 \quad (28)$$

You can convert the expression into a function of x, however by using the "unapply" command:

> $f := unapply(f, x)$;

$$f := x \to x^2 \quad (29)$$

> $f(5)$;

$$25 \quad (30)$$

You can also take the function $f = x^2$ and convert it back into an expression:

> $f := f(x)$;

$$f := x^2 \quad (31)$$

> $gt; subs(x = 5, f)$;

$$gt$$
$$25 \quad (32)$$

CTRL-S

1.2 ➡ Polynomials

The factorization of a polynomial, expansion of an expression, simplification of an expression, simplification of a rational expression and substitution into an expression can be dealt through direct MAPLE commands and context menus:

1.2.1 ➡ *Factorization of a Polynomial*

The command will show you full information about factorization on line by typing:

> ?factor

Command:

$$> factor(x^2+5\cdot x+6);$$

$$(x+3)\ (x+2) \tag{33}$$

Context Menu Result: You can use Maple's context menus to perform a wide variety of mathematical and other operations. Enter the polynomial and place your cursor on the last end of the polynomial or expression and right-click. The context menu offers several operations to choose from according to the expression that you are using. The above result through context menu is as under:

$$x^2+5\cdot x+6 \overset{\text{factor}}{=} (x+3)\ (x+2)$$

This result is obtained through right-click on the last end of the expression by selecting "Factor" on the context menu.

1.2.2 ➩ *Expansion of an Expression*

Command:

$$> expand(x^5\cdot(x^2+3\cdot x+1));$$

$$x^7+3\,x^6+x^5 \tag{34}$$

Context-Menu Result:

$$x^5\cdot(x^2+3\cdot x+1) \overset{\text{expand}}{=} x^7+3\,x^6+x^5$$

1.2.3 ➩ *Simplification of an Expression*

Command:

$$> simplify\left(4^{\left(\frac{1}{2}\right)}+\left(\frac{6}{4}\right)\right);$$

$$\frac{7}{2} \tag{35}$$

Context-Menu Reslt:

$$4^{\frac{1}{2}}+\frac{6}{4} \overset{\text{simplify constant}}{=} \frac{7}{2}$$

This result is obtained through Right-click on the last end of the expression by selecting "Simplify<Constant" on the context menu.

1.2.4 Simplification of a Rational Expression

Command:

$$simplify\left(\frac{(x+3)}{(x^2+5\cdot x+6)}\right);$$

$$\frac{1}{x+2} \quad (36)$$

Context-Menu Result:

$$\frac{(x+3)}{(x^2+5\cdot x+6)} \xrightarrow{\text{assuming integer}} \frac{1}{x+2}$$

1.2.5 Substitution into an Expression

Command:

$$> subs(x=y+3, 2\cdot x+5);$$

$$2y+11 \quad (37)$$

Context-Menu Result:

$$2\cdot x+5 \xrightarrow{\text{evaluate at point}} 2y+11$$

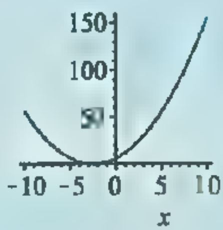
This result is obtained through Right-click on the last end of the expression by selecting "Evaluate at a point" on the context menu.

1.3 Graphics

1.3.1 Two-Dimensional Graph

Command:

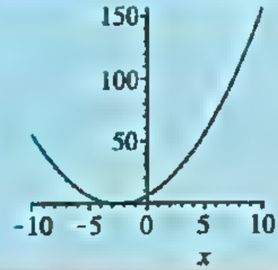
$$plot(x^2+5\cdot x+6);$$



Context-Menu Result:

x^2+5x+6

$$x^2+5x+6 \quad (38)$$

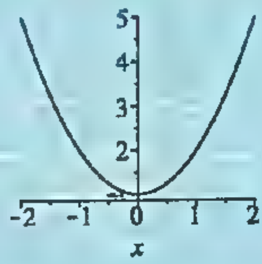
→ 

This result is obtained through Right-click on the last end of the expression by selecting "Plots < 2-D plot" on the context menu.

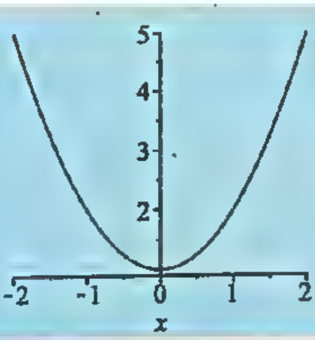
1.3.2 Two-Dimensional Plot With Domain and Range

Command:

```
plot( x^2 + 1, x=-2..2);
```



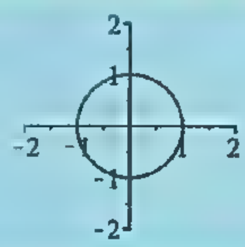
Context-Menu Result: Enter an expression or function, right-click on it and select Plots < Plot Builder < 2-D Plot and enter the domain for the expression or function. For example:

$x^2+1 \rightarrow$ 

1.3.3 Parameterized Form of a Function

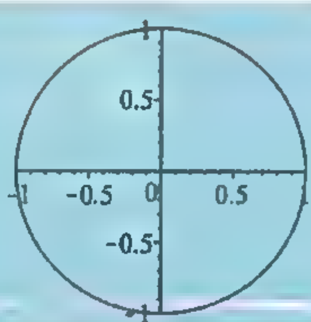
Command:

```
plot( [sin(t), cos(t), t=-4..4], -2..2, -2..2);
```



Context-Menu Result: Enter an expression or function, right-click on it and select Plots < Plot Builder <2-D parametric plot and enter the domain for the parameter t:

$\sin(t), \cos(t) \xrightarrow[t]{}$



1.4 Matrices

The command will show you full information about matrices on line by typing:

```
> ?matrices
```

1.4.1 Matrix and Vector Entry Arrangement

Command:

```
> with(linalg) :
> M := matrix(2, 2, [4, 3, 3, 2]);
```

$$M := \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \tag{39}$$

```
> N := matrix(2, 2, [6, 7, 8, 9]);
```

$$N := \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \tag{40}$$

Matrix addition:

```
> matadd(M, N);
```

$$\begin{bmatrix} 10 & 10 \\ 11 & 11 \end{bmatrix} \qquad (41)$$

Matrix Multiplication:

> *multiply*(*M*, *N*);

$$\begin{bmatrix} 48 & 55 \\ 34 & 39 \end{bmatrix} \qquad (42)$$

Transpose of a Matrix:

> *trans* := *transpose*(*M*);

$$trans := \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \qquad (43)$$

> *matadd*(*M*, *trans*);

$$\begin{bmatrix} 8 & 6 \\ 6 & 4 \end{bmatrix} \qquad (44)$$

Inverse of a Matrix:

> *inv* := *inverse*(*M*);

$$inv := \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \qquad (45)$$

> *multiply*(*M*, *inv*);

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (46)$$

Using Palettes: Use cursor button to select matrix palette. Click-"Matrix" <click-Choose (for the number of rows and columns of a required matrix) < click-Data type (to select integers entries of the rows and columns of a required matrix), then finally click-"Insert Matrix" and press ENTER key to obtain a required matrix:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad (47)$$

Matrix Addition:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix}$$

This result is obtained through context menu. Right -click on the last end of the matrix by selecting "Evaluate and display inline" to obtained the addition of two matrices.

Matrix Multiplication:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 13 & 16 \\ 29 & 36 \end{bmatrix}$$

This result is obtained through context menu. Right-click on the last end of the matrix by selecting "Evaluate and display inline" to obtained the product of two matrices.

Inverse of a Matrix:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{\text{inverse}} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

This result is obtained through context menu. Right-click on the last end of the matrix by selecting "Standard Operations" and then right-click on the "inverse" to obtained the inverse of a matrix.

Similarly,

Determinant of a Matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 3 & 5 & 1 \end{bmatrix} \xrightarrow{\text{determinant}}$$

Transpose of a Matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 3 & 5 & 1 \end{bmatrix} \xrightarrow{\text{transpose}} \begin{bmatrix} 1 & 4 & 3 \\ 2 & 2 & 5 \\ 3 & 6 & 1 \end{bmatrix}$$

1.5 MAPLE as Graphing-Calculator

You Click-start, then select-program <Maple 14 < click-"Maple Calculator" to obtain:

Maplesoft(TM) Graphing Calculator Overview

This graphically scientific calculator is available for use as part of your Maple(TM) installation or via a Web Server running MapleNet(TM). The calculator use Maple for calculations.

On toolsbar,

- use the "setting tab" to control the basic computation settings for the calculator.
- use the "Math tab" to select functions to apply, from basic functions to linear algebra to statistics.

- use the "Graph tab" to control over how graphs are displayed and what they display.
- use the "Data tab" to control the data used to produce a graph or the data you have tabulated directly.
- use the "variable tab" to control the variables you have assigned and their values .

To involve the graphing calculator, use your mouse to press the "Math tab" and select functions to apply. This will build up your expression for you in the input area, which is just below the session history area on the left side of the calculator. When you are ready to evaluate your expression, press ENTER key on your keyboard. Alternatively, you can press the "Graph button", to graph the expression, or the "data button", to tabulate values for the expression. For example,

1.5.1 **Differentiate $f(x) = x^2 + 4x + 4$ with respect to x at a point x = 2. The steps required for obtaining the graphing-calculator result are:**

Click-Math tab < Click-Calculus < Click-Differentiate to obtain:

Diff(1)

Cursor "1" requires A : the expression $x^2 + 4x + 4$, X: the differentiation of f(x) with respect to x and P: the differentiation of f(x) at a point x=2:

Diff(A,X,P)
Diff(x^2+4x+4,X,2)
Click-ENTER
8

1.5.2 **Integrate $f(x) = x^2 + 4x + 4$ with respect to x over the interval [0 1]. The graphing-calculator result is as under:**

int(1)
Cursor "1" requires A: the function $x^2 + 4x + 4$, X: the integration of f(x) with respect to x, P: the lower limit x=0 and Q: the upper limit x=1 of the integral:

int (A,X,P,Q)
int(x^2+4x+4,X,0,1)
Click-ENTER
6.333333

1.5.3 **Matrices:**

The steps required in entering a matrix on Graphing-Calculator are the following:

1. Click-Matrix/List-Editor, Click-Matrix<Click-Dimension (to choose the order of a matrix) < Click-List (to enter the rows and columns of a required matrix)
2. Click-Variables-Tab, Click-Matrx<Click-Clear Selection <Copy Selection <Click- Sort-List.
3. Click-Blank-Box (to denote the required matrix by A, say) <click-Save (to save the required matrix by A).
4. Click-Update (to display the rows and columns of a matrix A):

[[1,2],[3,4]].

5. Click-ENTER (to display the matrix A):

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

6. **Inverse of a Matrix:** Click-Math-Tab <Click-Linear Algebra <Click-Inverse (to obtain the inverse of a matrix A):

Inverse(A)
Click-ENTER

$$\begin{bmatrix} -2.000 & 1.000 \\ 1.500 & -0.500 \end{bmatrix}$$

7. **Determinant of a Matrix A:** Click-Math-Tab <Click-Linear-Algebra <Click-Determinant (to obtain the determinant of a matrix A):

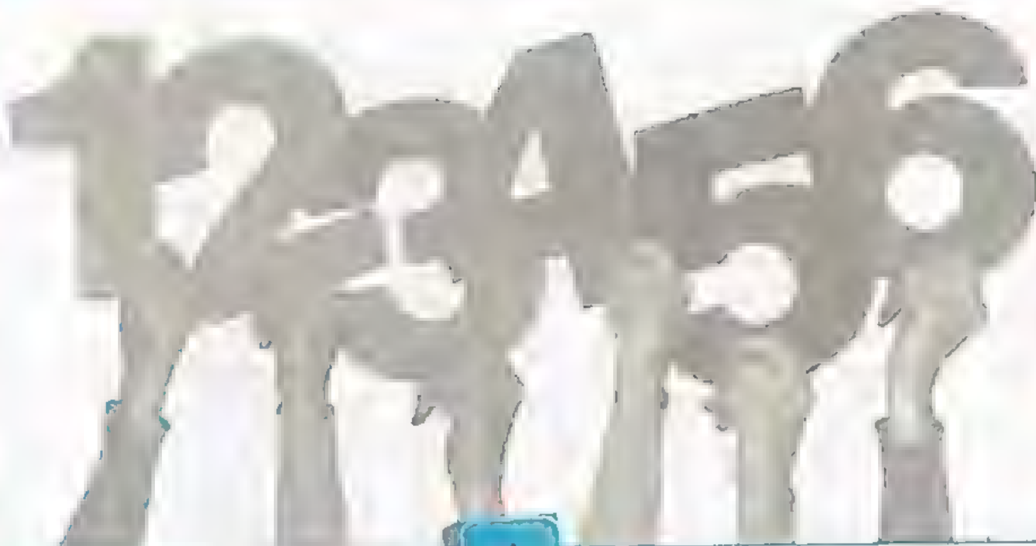
Determinant(A)
Click-ENTER
-2

8. **Transpose of a Matrix:** Click-Math-Tab <Click-Linear Algebra <Click-Transpose (to obtain the transpose of a matrix A):

Transpose(A)
Click-ENTER

$$\begin{bmatrix} 1.000 & 3.000 \\ 2.000 & 4.000 \end{bmatrix}$$

This Unit-I is typed under "MIXED-MODE" [Document mode + Worksheet mode].



Maple Based Examples of Mathematics-XII

Important Limits

Example 1: Use Maple commands to evaluate the limit of a function

a. $f(x) = x^2 + 2x + 2$, when x tends to 2.

b. $f(x) = \dfrac{(x^3 - a^3)}{(x-a)}$, when x tends to a.

If the command for the required limit of a function is not known to you, then, easily on line ,call the command by typing:

> *?limit*

This will show you all commands about the limits.

a. Command:

> $limit((x^2) + (2 \cdot x + 2), x = 2)$

$$10 \qquad (48)$$

Using Palettes: Use cursor button to select limit palette. Click-the required limit palette,and replace a by 2.Click (a+b) (for sum rule of a function), then press "Enter" key to obtain the required limit::

> $\lim_{x \to 2} ((x^2) + (2 \cdot x + 2))$

$$10 \qquad (49)$$

b. Command:

> $limit\left(\dfrac{(x^3 - a^3)}{(x-a)}, x = a\right);$

$$3a^2 \qquad (50)$$

Using Palettes: Use cursor button to select limit palette. Click-the required limit and replace a by a. Click-(a/b) (the quotient rule of a function), then "Enter" key to obtain the required limit:

$\lim_{x \to a} \dfrac{x^3 - a^3}{x - a}$

$$3a^2 \qquad (51)$$

Continuous and Discontinuous Functions

Example 2: Use Maple command iscont to test continuity of a function $f(x) = x^2+4$ at any point in an

a. interval from 0 to 1.

b. closed interval [0,1].

c. open interval (0,1).

a+b+c. Commands:

> $iscont(x^2 + 4, x = 0..1);$

$$true \qquad (52)$$

> $iscont(x^2 + 4, x = 0..1, 'closed');$

$$true \qquad (53)$$

> $iscont(x^2 + 4, x = 0..1, 'open');$

$$true \qquad (54)$$

Example 3: Use Maple commands to differentiate the following functions.

a. $f(x)=x^5+7x+2$ w.r.t variable x.

b. $f(x)=\dfrac{(x^4+2x+16)}{(x^3+3x-2)}$ w.r.t variable x.

c. $f(x)=(x^3+\sin(x)^2+arccosx$ w.r.t variable x.

d. $f(x)=x^2\cosh x+\operatorname{arcsinh} x$ w.r.t variable x.

a. Command:

$$> diff(x^5+7\cdot x+2, x);$$
$$5x^4+7 \qquad (55)$$

Context Menu:

$$> x^5+7\cdot x+2$$
$$> diff(x\char`^5+7*x+2, x)$$
$$5x^4+7 \qquad (56)$$

This result is obtained through right-click on the last end of the expression by selecting " Differentiate < x " on the context menu.

b. Command:

$$> diff\left(\frac{(x^4+2\cdot x+16)}{(x^3+3\cdot x-2)}, x\right);$$
$$\frac{4x^3+2}{x^3+3x-2}-\frac{(x^4+2x+16)(3x^2+3)}{(x^3+3x-2)^2} \qquad (57)$$

Context Menu:

$$> \frac{(x^4+2\cdot x+16)}{(x^3+3\cdot x-2)}$$
$$> diff((x\char`^4+2*x+16)/(x\char`^3+3*x-2), x)$$
$$\frac{4x^3+2}{x^3+3x-2}-\frac{(x^4+2x+16)(3x^2+3)}{(x^3+3x-2)^2} \qquad (58)$$

c. Command:

$$> diff(x^3+\sin(x)^2+\arccos(x), x);$$
$$3x^2+2\sin(x)\cos(x)-\frac{1}{\sqrt{1-x^2}} \qquad (59)$$

Context Menu:

$$> x\char`^3+\sin(x)\char`^2+\arccos(x)$$
$$> diff(x\char`^3+\sin(x)\char`^2+\arccos(x), x)$$
$$3x^2+2\sin(x)\cos(x)-\frac{1}{\sqrt{1-x^2}} \qquad (60)$$

d. Command:

> $diff(x^2 \cdot \cosh(x) + \operatorname{arcsinh}(x), x);$

$$2x\cosh(x) + x^2\sinh(x) + \frac{1}{\sqrt{1+x^2}} \qquad (61)$$

Context Menu:

> $x^2 \cdot \cosh(x) + \operatorname{arcsinh}(x)$

> $diff(x\text{^}2 * \cosh(x) + \operatorname{arcsinh}(x), x)$

$$2x\cosh(x) + x^2\sinh(x) + \frac{1}{\sqrt{1+x^2}} \qquad (62)$$

Example 4: Use Maple commands to find out the first and second derivative of a function $f(x) = x^4 + x^2 sinx + x + 2$ w.r.t x.

Command:

> $diff(x^4 + x^2 \cdot \sin(x) + x + 2, x, x);$

$$12x^2 + 2\sin(x) + 4x\cos(x) - x^2\sin(x) \qquad (63)$$

For second derivative, after command, press the "Enter" key two times to obtain the second derivative of a given functionabove result.

Context Menu:

> $x^4 + x^2 \cdot \sin(x) + x + 2$

> $diff(x\text{^}4 + x\text{^}2 * \sin(x) + x + 2, x)$

$$4x^3 + 2x\sin(x) + x^2\cos(x) + 1 \qquad (64)$$

> $diff((64), x)$

$$12x^2 + 2\sin(x) + 4x\cos(x) - x^2\sin(x) \qquad (65)$$

Result (18) is obtained through right click on the last end of the expression by selecting "Differentiate < x" on the context menu. For second derivative, click on the last end of the expression of result (18) by selecting again "Differentiate < x"to obtain the required second derivative. For derivatives higher than two, repeat the process again and again to obtain the required higher order derivatives.

Maclaurin's and Taylor's Expansions

Example 5: Use Maple commands to expand a function

a. $f(x) = e^x$ by Taylor's series expansion to fist four terms.

b. $f(x) = sinx$ by Taylor's series expansion to first 5 terms.

a. Command:

> $taylor(e^x, x = 0, 4);$

$$1 + \ln(e)\,x + \frac{1}{2}\ln(e)^2x^2 + \frac{1}{6}\ln(e)^3x^3 + O(x^4) \qquad (66)$$

Context Menu,

> e^x

> $series(\ e\text{^}x, x, 4)$

$$1 + \ln(e)\,x + \frac{1}{2}\ln(e)^2x^2 + \frac{1}{6}\ln(e)^3x^3 + O(x^4) \qquad (67)$$

This result is obtained through right click on the last end of the expression by selecting "Series < x" on the context menu.

b. Command:

> $taylor(\sin(x), x=0, 5);$

$$x - \frac{1}{6}x^3 + O(x^5) \quad (68)$$

Context Menu:

> $\sin(x)$
> $series(\sin(x), x, 5)$

$$x - \frac{1}{6}x^3 + O(x^5) \quad (69)$$

Maxima and Minima

Example 6: Use Maple commands to maximize and minimize a function

a. f(x)=cosx b. $f(x) = x^4 - 2x^2 + 3$ in the interval [-1,2].

a. Command:

> $maximize(\cos(x));$

$$1 \quad (70)$$

> $minimize(\cos(x));$

$$-1 \quad (71)$$

Context Menu.

> $\cos(x)$
> $Optimization[Maximize](\cos(x))$

$$[1., [x = 5.58237824894110\ 10^{-17}]] \quad (72)$$

This result is obtained through right click on the last end of the expression by selecting "Optimization < maximize local" on the context menu.

> $\cos(x)$
> $Optimization[Minimize](\cos(x))$

$$[-1., [x = 3.14159265358977]] \quad (73)$$

b. Command:

> $maximize(x^4 - 2x^2 + 3, x = -1..2);$

$$11 \quad (74)$$

> $minimize(x^4 - 2x^2 + 3, x = -1..2);$

$$2 \quad (75)$$

Differentiation of Vector Functions

Example 7: a. Find the derivatives of a vector function

a. r(t)=(sint,cost,sin2t) w.r.t variable t.

b. r(t)=(te^t,e^t/t,3e^t) w.r.t variable t.

a. Command:

```
> with(VectorCalculus) :
> r(t) := ⟨sin(t), cos(t), sin(2·t)⟩;
```

$$r := t \rightarrow VectorCalculus{:-}{<,>}(\sin(t), \cos(t), \sin(2\,t)) \quad (76)$$

```
> diff(r(t), t);
```

$$(\cos(t))e_x - \sin(t)e_y + 2\cos(2\,t)e_z \quad (77)$$

b. Command:

```
> with(VectorCalculus) :
```

$$> r(t) := \left\langle t\cdot\exp(t), \frac{\exp(t)}{t}, 3\cdot\exp(t)\right\rangle;$$

$$r := t \rightarrow VectorCalculus{:-}{<,>}\left(t\,e^t, e^t\frac{1}{t}, 3\,e^t\right) \quad (78)$$

(70)

```
> diff(r(t), t);
```

$$(e^t + t\,e^t)e_x + \left(\frac{e^t}{t} - \frac{e^t}{t^2}\right)e_y + 3\,e^t e_z \quad (79)$$

Integration

Example 8: Use Maple commands to evaluate the

a. indefinite integral of a function $f(x) = x^4 + x^3 + x^2 + x + 1$ w.r.t variable x.

b. definite integral of a function $f(x) = x^2$ w.r.t variable x.

c. definite integral of a function $f(x) = xe^x$ in the interval [0,1].

a. Command:

```
> int(x^4 + x^3 + x^2 + x + 1, x);
```

$$\frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x \quad (80)$$

Using Palettes: Use cursor button to select integral palette. Click-integral palette, insert the function required, then press "ENTER" key to obtain the integral of a given function:

$$> \int x^4 + x^3 + x^2 + x + 1\,dx$$

$$\frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x \quad (81)$$

b. Command:

```
> int(x^2, x=0..1);
```

$$\frac{1}{3} \quad (82)$$

Using Palettes:

> $\int_0^1 x^2\,dx$

$$\frac{1}{3} \quad (83)$$

c. Command:

> $int(x\cdot\exp(x), x=0..1);$

$$1 \quad (84)$$

Using Plalettes:

> $\int_0^1 x\cdot\exp(x)\,dx$

$$1 \quad (85)$$

Plane Analytic Geometry - Straight Line

Example 9: Find the distance between the two points A(1,2) and B(34) on two dimensional space 2D.

Before to start, watch the command on line by typing

> *?distance*

that will give you full detail of distance.

> *with(Student[Precalculus])* :

> *Distance* [1, 2], [3, 4]

$$2\sqrt{2} \quad (86)$$

or

> *?geometry[distance]*

> *with(geometry)* :

> *point* *A*, 1, 2 , *point*(*B*, 3, 4) :

> *distance*(*A*, *B*)

$$\sqrt{8} \quad (87)$$

Example 10: Find the equation of a line that passes through the two points A(1,1) and B(3,4) in 2D.

The command below will give you full detail of a line in 2D on line by typing

```
[> ?line

[> ?geometry[equation]
[> with(geometry) :
[> point(A, 1, 1), point(B, 3, 4) :
 > line(l, [A, B])
                                                              l                                  (88)
[> Equation(l, [x, y])
                                           1 − 3 x + 2 y = 0                                     (89)
[>
```

Example 11: a. Find the y-intercept of a line y=5x+3.

b. Find the y-intercept of a line y=5x+3, when x=0.

c. Find the x-intercept of a line y=5x−3, when y=0.

The command below will show you full detail of intercepts of a line in 2Don line by typing:

```
[> ?intercepts
```

a. Command:

```
[> with(student) :
 > intercept(y = 5 x + 4)
                                           {x = 0, y = 4}                                        (90)
```

b.

```
 > intercept(y = 5 x + 3, x = 0)
                                           {x = 0, y = 3}                                        (91)
```

c.

```
 > intercept(y = 5 x + 3, y = 0)
```

$$\left\{x = -\frac{3}{5}, y = 0\right\} \qquad (92)$$

Example 12: Slope-Intercept-Form. Find the equation of a line, if the slope of a line is -1 and y-intercept is Find also the x-intercept.

Command:

```
[> with(Student[Precalculus]) :
 > Line(-1, 2)
                                  y = -x + 2, -1, 2, 2                                           (93)
```

Example 13: Point-Slope-Form of a Line: Find the equation of a line that passes through the point A (2, 3) whose slope is 4. Find also the x-andy-intercepts of a line.

Command:

```
> with(Student[Precalculus]) :
> Line([2, 3], 4)
```

$$y=4x-5, 4, -5, \frac{5}{4} \quad (94)$$

Example 14: Two-Point-Form of a line: Find the equation of a line that passes through the two points A (1,1/2) and B(4,1). Find also the slope as well as the x- and y-intercepts of a line.
Command:

```
> with(Student[Precalculus]) :
```

$$> Line\left(\left[1, \frac{1}{2}\right], [4, 1]\right)$$

$$y=\frac{1}{6}x+\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, -2 \quad (95)$$

Conics-I
Example 15: Find the center and radius of a circle x^2+y^2+4x+6y=1.
The command below will show you full detail of a circle on line by typing:

```
> ?circle
> with(geometry) :
```

$$> circle(c, x^2+y^2+4\cdot x+6\cdot y=1);$$

$$c \quad (96)$$

```
> coordinates(center(c));
```

$$[-2, -3] \quad (97)$$

```
> radius(c);
```

$$\sqrt{14} \quad (98)$$

This command will reqire the x and y-coordinates on line.

Conics-II
Example 16:

a. First Parabola: Find the vertex, focus and directrix of a vertical parabola y^2-6x+2y+13=0.
b. Second Parabola: Find the equation of parabola when the vertex and focus are of the first parabola.
c. Third Parabola: Find the equation of parabola when the focus and directrix are of the first parabola.

The command below will show you full detail of parabola on line by typing:

```
> ?parabola
> with(geometry) :
```

$$> parabola(p1, y^2-6\cdot x+2\cdot y+13=0, [x, y])$$

$$p1 \quad (99)$$

```
> vertical(p1), coordinates(vertex(p1))
```

$$vertical(p1), [2, -1] \quad (100)$$

> focus(p1), coordinates(focus(p1))

$$focus\_p1, \left[\frac{7}{2}, -1\right] \quad (101)$$

> directrix(p1), Equation(directrix(p1))

$$directrix\_p1, x-\frac{1}{2}=0 \quad (102)$$

b.

> parabola(p2, ['vertex'=vertex(p1),'focus'=focus(p1)], [x, y]) :

> Equation(p2)

$$\frac{117}{4}+\frac{9}{4}y^2-\frac{27}{2}x+\frac{9}{2}y=0 \quad (103)$$

>

c.

> parabola(p3, ['focus'=focus(p1),'directrix'=directrix(p1)], [x, y]) :

> Equation(p3)

$$y^2-6x+2y+13=0 \quad (104)$$

Ellipse

Example 17:

a. First Ellipse: Find the center, foci, major axis and minor axis of an ellipse $2x^2+5y^2=10$.

b. Second Ellipse: Find the equation of an ellipse, when the foci and the length of the major axis are of the first ellipse.

c. Third ellipse: Find the equation of an ellipse, when the foci and the length of the minor axis are of the first ellipse.

The command below will show you full detail of ellipse on line by typing:

> ?ellipse

> with(geometry) :

> envHorizontalName :='x': EnvVerticalName :='y':

> ellipse(e1, $2\cdot x^2+y^2-4\cdot x+4\cdot y=0$) ;

> center(e1), coordinates(center(e1))

$$center\_e1, [1, -2] \quad (105)$$

> foci(e1), map(coordinates, foci(e1))

$$[foci\_1\_e1, foci\_2\_e1], [[1, -2-\sqrt{3}], [1, -2+\sqrt{3}]] \quad (106)$$

> MajorAxis(e1), MinorAxis(e1)

$$2\sqrt{6}, 2\sqrt{3} \quad (107)$$

b.

> ellipse(e2, ['foci'=foci(e1),'MajorAxis'=MajorAxis(e1)]) :

> detail(e2)

```
assume that the names of the horizontal and vertical axes are _x
and _y, respectively
```

name of the object	*e2*
form of the object	*ellipse2d*
center	$[1, -2]$
foci	$[[1, -2-\sqrt{3}], [1, -2+\sqrt{3}]]$
length of the major axis	$2\sqrt{6}$
length of the minor axis	$2\sqrt{3}$
equation of the ellipse	$96\_x^2 + 48\_y^2 - 192\_x + 192\_y = 0$

(108)

c.

```
> ellipse(e3, ['foci'=foci(e1),'MinorAxis'=MinorAxis(e1)]) :
> center(e2), coordinates(center(e2))
```
$$center\_e2, [1, -2] \qquad (109)$$
```
> Equation(e2)
```
$$96x^2 + 48y^2 - 192x + 192y = 0 \qquad (110)$$
```
> Equation(e3)
```
$$96x^2 + 48y^2 - 192x + 192y = 0 \qquad (111)$$

Hyperbola
Example 18:

a. First hyperbola: Find the center, foci, vertices and asymptotes of hyperbola $9y^2 - 4x^2 = 36$.
b. Second Hyperbola: Find the equation of hyperbola, when the foci and vertices are of the hyperbola.
c. Third Hyperbola: Find the equation of hyperbola, when the foci and the distance between the vertices are of the first hyperbola.

The command below will show you full detail of hyperbola on line by typing:

```
> ?hyperbola
```

a.

```
> with(geometry) :
> hyperbola(h1, 9·y^2 - 4·x^2 = 36, [x, y]) :
> center(h1), coordinates(center(h1))
```
$$center\_h1, [0, 0] \qquad (112)$$
```
> foci(h1), map(coordinates, foci(h1))
```
$$[foci\_1\_h1, foci\_2\_h1], [[0, -\sqrt{13}], [0, \sqrt{13}]] \qquad (113)$$
```
> vertices(h1), map(coordinates, vertices(h1))
```
$$[vertex\_1\_h1, vertex\_2\_h1], [[0, -2], [0, 2]] \qquad (114)$$
```
> asymptotes(h1), map(Equation, asymptotes(h1))
```
$$[asymptote\_1\_h1, asymptote\_2\_h1], \left[y + \frac{2}{3}x = 0, y - \frac{2}{3}x = 0\right] \qquad (115)$$

b.

```
> hyperbola(h2, ['vertices'=vertices(h1),'foci'=foci(h1)], [a, b]) :
> Equation(h2)
```
$$64a^2 + 576 - 144b^2 = 0 \qquad (116)$$

> *hyperbola*(*h3*, ['*foci*'= *foci*(*h1*),'*distancev*'= *distance*(*op*(*vertices*(*h1*)))], [*m*, *n*]) :
> *detail*(*h3*)

name of the object	*h3*	
form of the object	*hyperbola2d*	
center	[0, 0]	
foci	$[[0, -\sqrt{13}], [0, \sqrt{13}]]$	(117)
vertices	[[0, −2], [0, 2]]	
the asymptotes	$\left[n + \frac{2\,m}{3} = 0, n - \frac{2\,m}{3} = 0\right]$	
equation of the hyperbola	$64\,m^2 + 576 - 144\,n^2 = 0$	

$$64\,a^2 + 576 - 144\,b^2 = 0 \quad (118)$$

> *Equation*(*h3*)

$$64\,m^2 + 576 - 144\,n^2 = 0 \quad (119)$$

Differential Equations

Example 19: Find the general solution of a differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

The commond below will show you full detail of a differential equation on line by typing:

> *?differentialequations*

Choose "dsolve differential equations".

> $ode := diff(y(x), x) + \frac{x}{y(x)} = 0$

$$ode := \frac{\mathrm{d}}{\mathrm{d}x}\, y(x) + \frac{x}{y(x)} = 0 \quad (120)$$

> *dsolve*(*ode*)

$$y(x) = \sqrt{-x^2 + \_C1}, y(x) = -\sqrt{-x^2 + \_C1} \quad (121)$$

In "dsolve command", the derivative dy/dx is replaced by its derivative command: "diff(y(x),y)".

b. Find the general solution and particular solution of a differential equation dy/dx=x+y, y(0)= 1.

> $ode := diff(y(x), x) - x - y(x) = 0$

$$ode := \frac{\mathrm{d}}{\mathrm{d}x}\, y(x) - x - y(x) = 0 \quad (122)$$

> *dsolve*(*ode*)

$$y(x) = -1 - x + e^x\_C1 \quad (123)$$

> $ics := y(0) = 1$

$$ics := y(0) = 1 \quad (124)$$

> *dsolve*({ *ode*, *ics* })

$$y(x) = -1 - x + 2\,e^x \quad (125)$$

Orthogonal Trajectories

The general solution $y(x) = \operatorname{sqrt}\left[-x^2 + c_1\right]$ of the above problem "a" is the first family of furves (solution curves of a differential equation). This can also be written as $x^2 + y^2 = c$. The orthogonal trajectories of a first family of curves is the second family of curves represented by y=cx (family of homogeneous lines). The question is, how to view that the second family of curves is the orthogonal trajectories of the first family of curves?

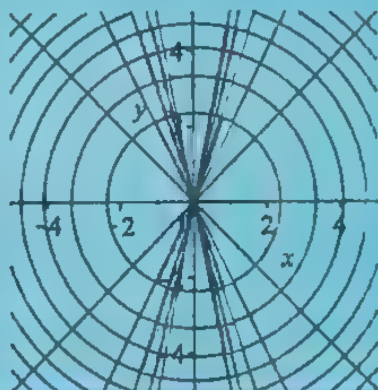
This can be viewed through command on line by typing:

> ?contourplot

> with(plots) :
> $F := contourplot(x^2 + y^2, x=-5..5, y=-5..5)$

$F := PLOT(...)$ **(126)**

> $G := plot(\{seq(c \cdot x, c=-5..5)\}, x=-5..5, y=-5..5)$:
> display({F, G});



Partial Differentiation

Example 20 : Find partial derivatives of the following Functions:

a. $f(x, y) = x^3 + y^3 + 3xy^2 + 4x^2y$ w.r.t variables x and y.

b. $f(x, y) = y\sin x + x\cos y + x^2$ w.r.t variables x and y.

a.

Command:

> $\operatorname{diff}(x^3 + y^3 + 3 \cdot x \cdot y^2 + 4 \cdot x^2 y, x)$;

$3x^2 + 3y^2 + 8xy$ **(127)**

> $\operatorname{diff}(x^3 + y^3 + 3 \cdot x \cdot y^2 + 4 \cdot x^2 y, y)$;

$3y^2 + 6xy + 4x^2$ **(128)**

b.

> $diff(y \cdot \sin(x) + x\cos(y) + x^2, x)$;

$y\cos(x) + \cos(y) + 2x$ **(129)**

> $diff(y\cdot\sin(x) + x\cdot\cos(y) + x^2, y);$

$$\sin(x) - x\sin(y) \quad (130)$$

>

Using Palettes: Use cursor button to select expression in which you are interested. In this problem, the expression is partial derivative palette. Click-partial derivative palette, insert the given function, then press "ENTER" key to obtain the partial derivatives of a given function:

> $\frac{\partial}{\partial x}(x^3 + y^3 + 3\cdot x\cdot y^2 + 4x^2\cdot y)$

$$3x^2 + 3y^2 + 8xy \quad (131)$$

> $\frac{\partial}{\partial y}(x^3 + y^3 + 3\cdot x\cdot y^2 + 4x^2\cdot y)$

$$3y^2 + 6xy + 4x^2 \quad (132)$$

Introduction to Numerical Methods

Example 21: Find the aproximates roots of the nonlinear equations

a. $x^2 - 5x + 6 = 0$ with initial start $x_0 = 1.8$.

b. $sinx - e^{(-x)} = 0$ with initial start $x_0 = 0.5$.

The command below will show you full detail of the approximate root of linear and non-linear equations on line by typing without initial start:

> ?fsolve

> $fsolve(x^2 - 5\cdot x + 6)$

$$2.000000000, 3.000000000 \quad (133)$$

The quadratic function f(x) is also a second degree polynomial. The numerical solution through polynomial is:

> $Polynomial := x^2 - 5\cdot x + 6$

$$Polynomial := x^2 - 5x + 6 \quad (134)$$

> $fsolve(Polynoinial)$

$$2.000000000, 3.000000000 \quad (135)$$

b.

> $fsolve(\sin(x) - \exp(-x))$

$$0.5885327440 \quad (136)$$

Context Menu:

> $\sin(x) - \exp(-x)$

> $fsolve(\sin(x) - \exp(-x))$

$$0.5885327440 \quad (137)$$

This result is obtained through right-click on the last end of the expession by selecting"Solve < Numerically Solve" on the context menu.

Numerical Quadrature

Example 21: Approximate the integral $\int_0^1 e^{(-x)}\,dx$ in the interval [0,1] by

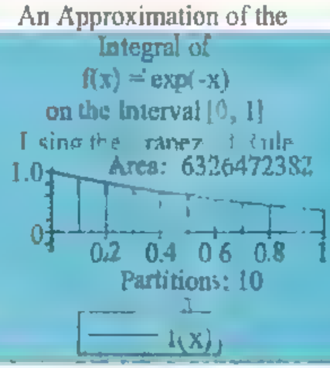
a. Trapezoidal rule. b. Simpson rule

a. The command below will show you full detail about Trapezoidal rule on line by typing:

> *?trapezoid*

> *with(Student[Calculus1]) :*

> *ApproximateInt*(exp(−x), x = 0 .. 1, *method* = *trapezoid*, *output* = *plot*);

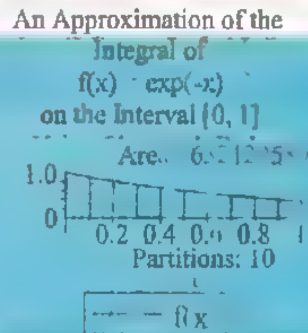


b. The command below will show you full detail about Simpson rule on line by typing:

> *?simpson*

> *with(Student[Calculus1]) :*

> *ApproximateInt*(exp(−x), x = 0 .. 1, *method* = *simpson*, *output* = *plot*);



The actual value of the intergral $\int_0^1 e^{(-x)}\,dx$ is:

> *int*(exp(−x), x = 0 .. 1);

$$1 - e^{-1} \tag{138}$$

Use calculator to simplify result (91) and then compare to Trapezoidal and Simpson results.

(139)

UNIT 1

FUNCTIONS AND LIMITS

This unit tells us how to :

- identify the domain and range of a functions through graphs.
- draw the graph of modulus function and identify its domain and range.
- recognize the composition of a function and then to find out the composition of two functions.
- describe the inverse of a function and then to find out the inverse of composition of two functions.
- recognize the algebraic and transcendental functions as well as the concepts of explicit, implicit and parametric functions.
- display graphically the explicit, implicit and parametric functions as well as the compound functions.
- introduce the limit of a function with respect to real number intervals on the real number line, the open and closed intervals and its location on a real number line.
- explain the meaning of x tends to zero, x tends to a and x tends to infinity.
- define the limit of a sequence when the limit of a sequence with n-th term is given.
- define the limit of a function and the statement of theorems on limits of sum, difference, product and quotient of functions.
- evaluate the limits of a function in case of some special functions.
- evaluate the limits of algebraic, exponential and trigonometric functions.
- introduce the continuous and discontinuous functions.
- recognize the left hand and right hand limits through examples.
- define the continuity of a function at a point and in an interval.
- test the continuity and discontinuity of a function at a point and in an interval.

1.1 Functions

Whether you realize it or not, you use the concept of a function frequently in your everyday life. The cost of a car trip is a function of the cost of gasoline. The acceleration of an airplane on takeoff is a function of the thrust of its engines. The depth of field in a photograph is a function of the lenses opening.

In each of these situations, we are saying that the first thing is dependent on the second independent. For example, as the price of gasoline goes up (or down), it costs more (or less) to run a car. However, the total cost of running a car is affected by other variables, not just by the price of fuel.

In mathematics, the word function is used in much the same way, but more restrictively. Thus, the mathematical function demands that one quantity is uniquely determined by one or more quantities.

Definition 1.1.1: [Function]: A function $y=f(x)$ is a rule that assigns for each value of the independent variable (input) **x** a unique value of the dependent variable (output) **y**:

$$y = f(x) \qquad (1)$$

Example 1.1.1: A developer estimates that the total cost on construction of x large number ofisport complexes in each provincial capital is approximated by

$C(x) = x^2 - 80x + 80$,

where C(x) is the cost in hundred thousands of dollars with respect to x number of sports complexes. Find the cost of 4 sports complexes.

Solution: The cost of 4 complexes is obtained by putting x=4 in the given cost function:

$C(4) = 4^2 + 80(4) + 80 = 416$ Hundred Thousands of dollars

Definition 1.1.2: [Domain and Range]: Let $y=f(x)$ be a function of independent variable x. The set of real numbers that can be substituted for the independent variable x and give real numbers for the dependent variable y is called the domain of the function. The set of real numbers obtained for the dependent variable y is called the range of a function.

Example 1.1.2:[Domain and Range]: Determine the domain and range of a function $f(x) = x^2 + 5$.

Solution: The given function f(x) with the replacement of the independent variable x with any real number is giving the real value. Thus, the domain set of a function f(x) is $D = \{x \mid x \in R\}$.

For range value of a function, the function f(x) with substitution x= –3 are:

$f(-3) = (-3)^2 + 5 = 9 + 5 = 14$

The squaring of a negative number in a given function will yield a positive product. Thus, the functional value will always be five greater than the number squared. The least possible value of the function is 5, which occurs when $x=0$. Thus, the range set of a function f(x) is $R = \{f(x) \mid f(x) \geq 5\}$.

i) Identification of the domain and range from the graph of a function

Consider the graph of a function

$$y=\frac{1}{x-2}$$

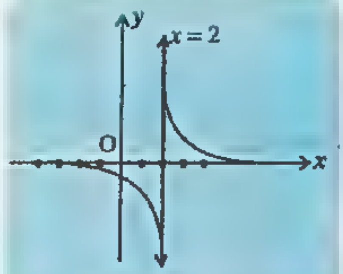


Fig. 1.1

To determine the domain of the function $y=1/(x-2)$, ask the following question "What real numbers may be substituted for x which will not give real values for y?" An obvious answer is $x=2$. For $x = 2$, the denominator would be zero and we know that division by zero is undefined. Since 2 is the only restriction. State the domain in a negative sense, $D=\{x \mid x\neq 2\}$. This statement implies that all real numbers, except 2, may be substituted for x to obtain real values for y.

Are there any restrictions on the range? Can y be equal to all real numbers? By just looking at the function, it is difficult to tell. However, by solving for x, the solution becomes apparent:

$$y=\frac{1}{x-2}$$
$$(x-2)y=1$$
$$xy-2y=1$$
$$xy=2y+1$$
$$x=\frac{2y+1}{y}$$

From this result, we can see that we could substitute any real number, except zero for y and obtain a real number for x. Thus, the range set is $R=\{y \mid y\neq 0\}$.

Example 1.1.3:[Domain and Range]: The formula P=64d is used to determine the pressure P on objects that are immersed in saltwater, d feet below a free surface. Determine the domain D and the range R.

Solution: For the function P=64d, the depth d is measured as a positive number of feet. Thus, the domain set is $D=\{d \mid d\geq 0\}$. Since d is nonnegative, P must be nonnegative, and the range set is then $R=\{P \mid P\geq 0\}$.

Definition 1.1.3:[Compound Function]: A function that defined by more than one equation is called a compound function. For example, a single equation may define the rate of pay for the 'ay shift (7 A.M. to 3 P.M.). Assume that the rate of pay for the day shift is modeled by equation is P(r)=8r, where r is the hourly rate. Another equation may define the rate of pay for the evening shift (3:30 P.M. to 11:30 P.M.).

The equation for this shift is P(r)=8(r+50). We can combine these two equations to create a compound function:

$$[illegible] \begin{cases} 8r, & \text{day shift} \\ [illegible](r+5[illegible]), & [illegible] \end{cases}$$

[illegible] (Compound Function): E[illegible]aluate the co[illegible]pound function

$$[illegible] \begin{cases} [illegible] & [illegible] \le 1 \\ x^2-1, & \text{for } x>1 \end{cases}$$

for the [illegible]dependent variables:

a. $x=-1$ b. $x=1$ c. $x=2$

Solution:

a. Since $x=-1$, which is less than 1, use the function $f(x)=3x+2$ to obtain:

$f(-1)=3(-1)+2=-1$

b. Since $x=1$, use the function $f(x)=3x+2$ to obtain:

$f(1)=3(1)+2=5$

c. Since $x=2$, which is greater than 1, use the function $f(x)=x^2-1$ to obtain:

$f(2)=(2)^2-1=4-1=3$

[illegible] and range

Graph the following absolute-valued functions and identify its domain and range:

a. $f(x)=|x|$ b. $f(x)=|3x+4|$

a. F[illegible] the definition of absolute value function, the given function is

$$f(x)=\begin{cases} x, & \text{if } x\ge 0 \\ -x, & \text{if } x<0 \end{cases}$$

The graph will consist of portions of the two lines with equations $y=x$ and $y=-x$. For $x\ge 0$, the graph is a line $y=x$, and for $x<0$, the graph is a line $y=-x$. The two partial lines meet at (0,0). Use the tabular form to obtain the graph of a function:

[illegible] $x\ge 0$			
x	[illegible]	1	2
y	[illegible]	1	[illegible]

$f(x)=-x,\ x\le 0$			
x:	0	-1	-2
y:	0	1	2

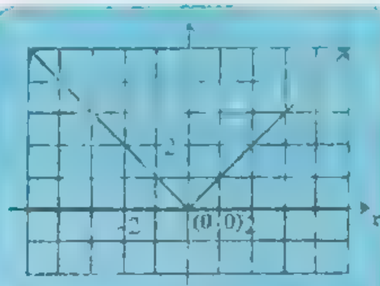


Fig. 1.2

From the tabular form of a function we noted that the domain set is $(-\infty,\infty)$ and the range set is $[0,\infty)$.

b. From the definition of absolute value function, the given function is

$$f(x)=\begin{cases}3x+4, & if\ 3x+4\geq 0\\ -(3x+4), & if\ 3x+4<0\end{cases}$$

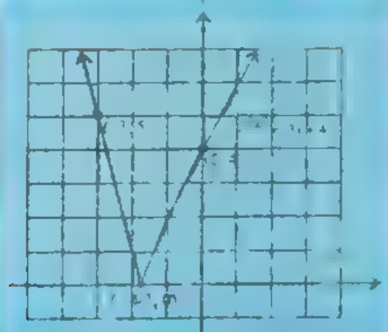


Fig. 1.3

The inequality $3x+4\geq 0$ is satisfied whenever $x\geq -\frac{4}{3}$, and $3x+4<0$ is satisfied whenever $x<-4/3$. If $x=-4/3$, $y=0$, so the graph will consist of two lines that meet at $(-4/3,0)$. Use the tabular form to obtain the graph of a function:

$f(x)=3x+4,\ 3x+4\geq 0$		
x:	−4/3	0
y:	0	4

$f(x)=-(3x+4),\ 3x+4<0$		
x:	−4/3	−3
y:	0	5

This function also has a domain set $(-\infty,\infty)$ and range set is $[0,\infty)$.

Definition 1.1.4: [One–To–One Functions]: A function $f(x)$ is said to be one–to–one if each range value corresponds to exactly one domain value, otherwise many–to–one function, if each range value corresponds to more than one domain values.

Example 1.1.5: Are the following relations are one–to–one or many–to one:

a. $y=x^2$ b. $y=x^3$ c. $y^2=x$

Solution:

a. The given function is many–to–one, since the value of $y=4$ corresponds to two values of $x=2,-2$.

b. The given function is one–to–one, since the value of $y=8$ corresponds to exactly one value of $x=2$.

c. The given equation is not a function, because two values of y correspond to one value of x will not constitute a function.

1.2 Composition of Functions

i) Recognition of composition functions

Many of the most useful functions for applications are created by combining simpler functions. Viewing complex functions as combinations of simpler functions often makes them easier to understand and use.

The term "**composition of functions**" (or "**composite function**") refers to the combining of functions in a manner where the output from one function becomes the input for the next function.

In mathematical terms, the range (the y-value answers) of one function becomes the domain (the x-values) of the next function.

Definition 1.2.1: [Composite Function]: If $f(x)$ and $g(x)$ are the two functions, then, the **composite function or composition of** f and g leads the notations:

$(f\circ g)(x)=f(g(x))$ (2-a)

This means "f composed with g of x" or "f of g of x ".

$(g\circ f)(x)=g(f(x))$ (2-b)

This means " g composed with f of x" or" g of f of x ".

Example 1.2.1: [Composite Functions]: Let $f(x)=2x-1$ and $g(x)=\sqrt{3x+5}$. Find each of the following:

a. $g(f(4))$ b. $f(g(4))$ c. $f(g(-2))$

Solution:

a. The function $f(x)$ for $x=4$ is used to obtain $f(4)=2(4)-1=7$. Use $f(4)$ in equation (2-b) to obtain:

$g(f(4)) = g(7) = \sqrt{3(7)+5} = \sqrt{26}$

b. The function g(x) for $x=4$ is used to obtain $g(4) = \sqrt{3(4)+5} = \sqrt{17}$. Use g(4) in equation (2-a) to obtain:

$f(g(4)) = f(\sqrt{17}) = 2\sqrt{17} - 1$

c. $f(g(-2))$ does not exist, since -2 is not in the domain of g.

ii) Composition of two given functions

Let $f(x) = 4x+1$ and $g(x) = 2x^2+5x$. Find each of the following:

(a). $g(f(x))$ (b). $f(g(x))$

Solution:

a. Using the given functions to obtain:

$$g(f(x)) = g(4x+1), \quad f(x) = 4x+1$$
$$= 2(4x+1)^2 + 5(4x+1)$$
$$= 2(16x^2+8x+1) + 20x+5$$
$$= 32x^2+16x+2+20x+5$$
$$= 32x^2+36x+7$$

b. Using the given functions to obtain:

$$f(g(x)) = f(2x^2+5x), \quad g(x) = 2x^2+5x$$
$$= 4(2x^2+5x)+1$$
$$= 8x^2+20x+1$$

This example shows that $f(g(x))$ is not usually equal to $g(f(x))$.

Example 1.2.2:[Composite Function]: Air pollution is a problem for many metropolitan areas. Suppose that carbon monoxide is measured as a function of the number of people according to the following information:

Number of People	Daily Carbon Monoxide Level (in parts per million)
100,000	1.41
200,000	1.83
300,000	2.43
400,000	3.05
500,000	3.72

Further studies show that a refined formula for the average daily level of carbon monoxide in the air is

$$L(p)=0.7\sqrt{p^2+3}$$

Further assume that the population of a given metropolitan area is growing according to the formula $p(t)=1+0.02t^3$, where t is the time from now (in years) and p is the population (in hundred thousands). Based on these assumptions, what level of air pollution should be expected in 4 years?

Solution: The level of pollution at time t is given by the composite function:

$$L(p(t))=L(1+0.02t^3)$$
$$=0.7\sqrt{(1+0.02t^3)^2+3}, \quad p(t)=1+0.02t^3 \qquad (3)$$

The air pollution expected in 4 years is obtained by putting t=4 in equation (3):

$$L(p(t))=L(1+0.02t^3)$$
$$=0.7\sqrt{[1+0.02(4)^3]^2+3}\approx 2.0 \text{ ppm}$$

3 Inverse of Composition Functions

Definition 1.3.1: [Inverse Functions]: Let y=f(x) be a function of x. This function takes a dependent variable y in response of independent variable x. The function that takes x as dependent variable in response of y as the independent is then called the **inverse function** of $f(x)$ and is denoted by:

$$x=f^{-1}(y) \qquad (4)$$

The symbol f^{-1}(y) means the inverse of f and does not mean $\frac{1}{f}$

For example, if y=f(x) is one–to–one function, then the inverse of y=f(x) is the function $x=f^{-1}(y)$ formed by interchanging the independent and dependent variables x and y for y=f(x). Thus, if (a, b) is a point on the graph of $f(x)$, then (b, a) will be a point on the graph of the inverse of $f(x)$. The domain and range of y=f(x) are also valid for its inverse function $x=f^{-1}(y)$.

Note that, if $f(x)$ is not one–to–one, then $f(x)$ does not have an inverse function.

Description of the inverse of composition of two given functions

Example 1.3.1:[Inverse Function]: Find the inverse function of f(t)=3t–8.

Solution: The function $f(t)$ takes an output 3t–8 in response of input t. The inverse function must take an output t in response of input 3t–8:

$$f^{-1}(3t-8)=t \qquad (5)$$

If z=3t–8 say, then t = (z+8)/3. Use these values in equation (5) to obtain:

$$f^{-1}(z)=\frac{z+8}{3}$$

Put t as its argument instead of z to obtain the inverse function of $f(t)=3t-8$:

$$f^{-1}(t)=\frac{t+8}{3}$$

Example 1.3.2:[Inverse Function]: Let $f(x)=2x+3$ and $g(x)=3x$ and $h(x)=f[g(x)]$. Write expressions for the following functions

a. $h(x)$ b. $f^{-1}(x)$ c. $g^{-1}(x)$ d. $h^{-1}(x)$

Solution:

a. In response of $f(x)$ and $g(x)$, the function $h(x)$ is:

$$h(x)=f[g(x)]$$
$$=f(3x),\ g(x)=3x$$
$$=2(3x)+3$$
$$=6x+3$$

b. In response of $f(x)$, the inverse of $f(x)$ is:

$$x=f^{-1}(2x+3)$$
$$x=f^{-1}(z), \qquad z=2x+3 \Rightarrow x=(z-3)/2$$
$$\frac{z-3}{2}=f^{-1}(z)$$
$$\frac{x-3}{2}=f^{-1}(x) \qquad \textit{insert } x \textit{ instead of } z$$

c. In response of $g(x)$, the inverse of $g(x)$ is:

$$x=g^{-1}(3x)$$
$$x=g^{-1}(z), \qquad z=3x \Rightarrow x=\frac{z}{3}$$

$$\frac{z}{3} = g^{-1}(z)$$

$$\frac{x}{3} = g^{-1}(x) \qquad \textit{insert x instead of z}$$

d. In response of h(x), the inverse of h(x) is:

$$x = h^{-1}(6x+3)$$

$$x = h^{-1}(z), \qquad z = 6x+3 \Rightarrow x = \frac{(z-3)}{6}$$

$$\frac{z-3}{6} = h^{-1}(z)$$

$$\frac{x-3}{6} = h^{-1}(x) \qquad \textit{insert x instead of z}$$

Example 1.3.3: [Inverse Functions]: Let $f(x) = 4x+1$ and $g(x) = 2x^2 + 5x$. Find the inverse of the following functions:

a. g[f(x)] b. f [g(x)]

Solution:

a. Using the given functions to obtain:

$$\begin{aligned} g[f(x)] &= g[4x+1] \\ &= 2(4x+1)^2 + 5(4x+1) \\ &= 2(16x^2+8x+1) + 20x+5 \\ &= 32x^2+16x+2+20x+5 \\ &= 32x^2+36x+7 \end{aligned}$$

The function g[f(x)] does not have an inverse, since g[f(x)] is not one–to–one (many–to–one) function.

b. Using the given functions to obtain:

$$\begin{aligned} f[g(x)] &= f[2x^2+5x] \\ &= 4(2x^2+5x)+1 \\ &= 8x^2+20x+1 \end{aligned}$$

The function f[g(x)] does not have an inverse, since f[g(x)] is not one–to–one (many–to–one) function.

Exercise 1.1

1. Identify the independent and dependent variables for the following problems:

a. P=64d b. $F(c)=\frac{9}{5}c+32$

c. $C(F)=\frac{5}{9}(F-32)$ d. $s=f(r,\theta)=r.\theta$

e. $F=\theta(m,a)=ma$ f. $SA=\theta(l,w,h)=2lw+2lh+2wh$

2. Evaluate the following functions for the indicated independent variables:

a. $f(x)=3x^2+7x-5;\ f(3), f(-4), f(a+h)$

b. $f(t)=\frac{t+5}{t-3},\ f(2), f(7.4), f(-3.7)$

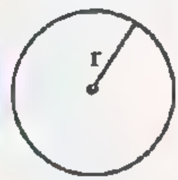
c. $g(R)=\frac{R^2-R+6}{R-3},\ g(2), g(3), g(\frac{3}{8})$

d. $f(t)=3t^2+2t-\sqrt{t},\ f(3.217), f(5.613), f(\pi)$

3. The circumference of a circle is given by $C(r)=2\pi r$, where r is the length of the radius (see the graph). Find

a. C(2.34 in) b. C(6.41 m) c. $C\left(\frac{5}{11}in\right)$

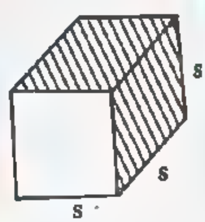
4. The area of a circle is given by $A(r)=\pi r^2$, where r is the length of the radius (see the graph). Find:



a. A (2.34 in.) b. A(6.41 m)

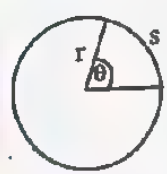
c. $A\left(\frac{5}{11}in.\right)$

5. The total surface area of a cube is given by the function $f(s)=6s^2$, where s is the length of the side of the cube (see the graph).Find:



a. f(3.75 m) b. f(6.05 in) c. f(13.42 mm)

6. The measure of the angle θ in radians is given by $\theta=f(s,r)=\frac{s}{r}$, where s' is the length of the arc



determined by $\angle\theta$ and r is the length of the radius of the circle (see the graph). Find:

a. $f(4.71, 3)$ b. $f(15.71,5)$

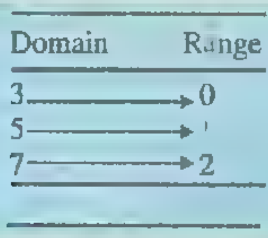
7. If $f(x)=\begin{cases} x-3, & for\ x<0 \\ 2x+5, & for\ x\geq 0 \end{cases}$, then find out

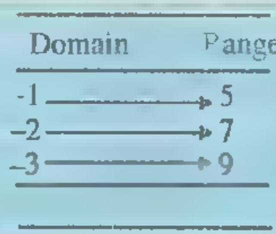
a. $f(-1)$ b. $f(0)$ c. $f(1)$

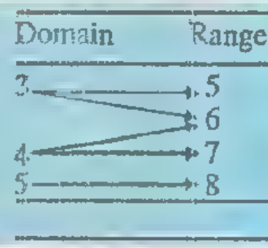
8. If $f(x)=\begin{cases} \dfrac{x^2-4}{x-2}, & for\ x\neq 2 \\ 4, & for\ x=2 \end{cases}$, then find out

a. $f(0)$ b. $f(2)$ c. $f(4)$

9. Indicate whether each table specifies a function y=f(x):

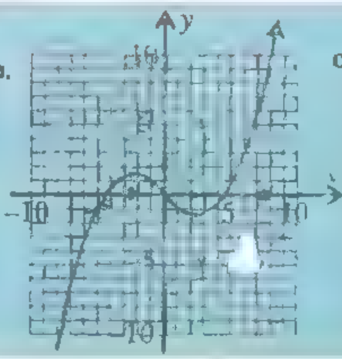
a.


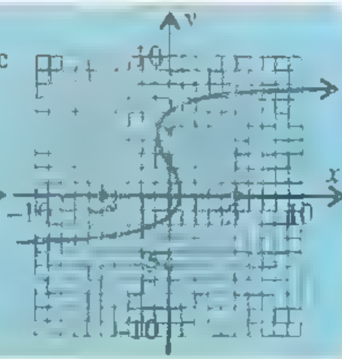
b.


c.


10. Indicate whether each graph specifies a function y=f(x):

a.


b.


c


11. Determine the domain and range of the following functions:

a. $y=3x+4$ b. $f(t)=t^2+5$ c. $SA=f(r)=4\pi r^2$

12. Find the composite functions f[g(x)] and g[f(x)] of the following functions:

a. $f(x)=x^2+1,\ g(x)=2x$ b. $f(x)=\sin x,\ g(x)=1-x^2$

c. $f(t)=\sqrt{t},\ g(t)=t^2$ d. $f(u)=\dfrac{u-1}{u+1},\ g(u)=\dfrac{u+1}{1-u}$

e. $f(x)=\sin x,\ g(x)=2x+3$ f. $f(x)=\frac{1}{x},\ g(x)=\tan x$

13. Determine the inverse function of each of the following functions:

a. $y=f(x)=x+5$ b. $y=f(x)=2x+7$

c. $y=f(x)=2(x-4)$ d. $y=f(x)=\frac{x+4}{2}$

14. Graph each of the following absolute functions:

a. $y=|x-4|$ b. $y=|-4-x|$

c. $y=|2x+5|$ d. $y=-|x|$

1.4 Transcendental Functions

A polynomial $P_n(x)$ is a function of the form

$$f(x)=P_n(x)=a_n x^n+a_{n-1}x^{n-1}+....+a_2x^2+a_1x+a_0 \qquad (6)$$

with n is a nonnegative integer and $a_n, a_{n-1},...,a_2,a_1,a_0$ are constants. If $a_n\neq 0$, then, the integer n is called the **degree** of the polynomial. The constant a_n is called the **leading coefficient** and the constant a_0 is called the **constant term** of the polynomial function. In particular, the polynomial (6) is going to be a

constant function by putting n=0: $f(x)=a_0$

linear function by putting n=1: $f(x)=a_1x+a_0$

quadratic function by putting n=2: $f(x)=a_2x^2+a_1x+a_0$

cubic function by putting n=3: $f(x)=a_3x^3+a_2x^2+a_1x+a_0$

Algebraic Function: A function $f(x)$ is called algebraic if it can be constructed using algebraic operations (such as adding, subtracting, multiplying, dividing, or taking roots) starting with polynomials. Any rational function is an algebraic function.

Transcendental Function: Functions that are not algebraic are called transcendental. The functions, such as all trigonometric functions, hyperbolic functions, exponential functions and logarithmic functions are called transcendental functions.

Trigonometric Functions: Trigonometric functions are the functions sine, cosine, tangent, secant, cosecant and cotangent. The word trigonometry is derived from Greek words which mean triangle measurement. Although trigonometry has been organized into an abstract mathematical system, much of it is still concerned with triangle measurement. This is completely discussed in XI class course.

Exponential Functions: The exponential function has widespread application in many areas of science and engineering. Areas which utilize the exponential function include expansion of materials, laws of cooling, radioactive decay and the discharge of a capacitor.

Definition 1.4.1: [Exponential Function]: An equation of the form

$f(x)=b^x, \quad b>0, b\neq 1$, b is a positive constant,

defines an exponential function for each different constant b, called the base. The domain of $f(x)$ is the set of all real numbers, and the range of $f(x)$ is the set of all positive real numbers.

We require the base to be positive and to avoid imaginary numbers such as $(-2)^{1/2}=\sqrt{-2}=i\sqrt{2}$. We conclude b=1 as a base, since $f(x)=1^x=1$ is a constant function.

The properties of exponential function are summarized in the box:

Exponent Laws : If a and b are positive real numbers, $a\neq 1, b\neq 1$, then,

1. $a^x a^y = a^{x+y}, \frac{a^x}{a^y}=a^{x-y}, (a^x)^y = a^{xy}, (ab)^x = a^x b^x, \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
2. $a^x = a^y$ if and only if $x=y$
3. For $x\neq 0$, $a^x=b^x$, if and only if a=b

Base e Exponential Functions: Of all possible bases b, it can use for the exponential function $y=b^x$, which ones are the most useful? If you look at the keys on a scientific calculator, you will likely see 10^x and e^x. It is clear why base 10 would be important, because our number system is a base 10 system. But what is e, and why is it included as a base? It turns out that base e is used more frequently than all other bases combined. The reason for this is that certain formulas and the results of certain processes found in calculus and more advanced mathematics take on their simplest form if this base is used. This is why you will see e used extensively in expressions and formulas that model real–world phenomena. In fact, its use is so prevalent that you will often hear people refer to $y=e^x$ as the exponential function.

The base e is an irrational number (like π) it cannot be represented exactly by any finite decimal fraction. However, e can be approximated as closely as we like by evaluating the expression

$$\left(1+\frac{1}{x}\right)^x \qquad (7)$$

for sufficiently large x. What happens to the value of expression (7) as x increases without bound? The results are summarized in the following table:

x	$\left(1+\frac{1}{x}\right)^x$
1	2
10	2.59374...
100	2.70481...
1000	2.71692...
10000	2.71814...
100000	2.71827...
1000000	2.71828...

Interestingly, the value of expression (7) is never close to 1, but seems to be approaching a number close to 2.7183. In fact, as x increases without bound, the value of expression (7) approaches an irrational number that we call e. The irrational number e to twelve decimal places is:

e=2.718 281 828 459

Growth and Decay Applications: Most exponential growth and decay problems are modeled using base e exponential functions. The problems related to growth and decay are as under.

Example 1.4.1: [Exponential Growth]: Cholera, an intestinal disease, is caused by a cholera bacterium that multiplies exponentially by cell division as given approximately by

$$N = N_0 e^{1.386t}$$

with N is the number of bacteria present after t hours and N_0 is the number of bacteria present at the start (t=0). If we start with 25 bacteria, how many bacteria (to the nearest unit) will be present in

a. 1 hour? b. 3 hours? c. 4 hours? d. Interpret

Solution: Use the amount of initial bacteria $N_0 = 25$ in the given equation to obtain:

$$N = 25e^{1.386t}, \quad N_0 = 25 \qquad (8)$$

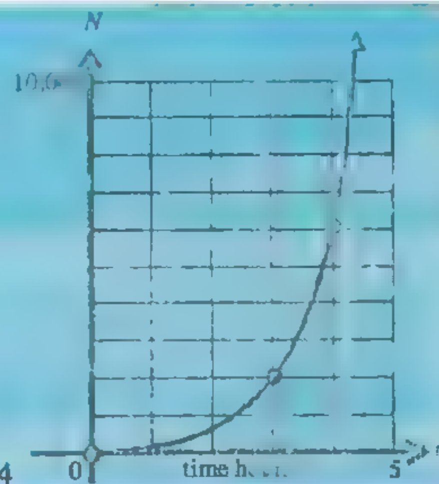


Fig. 1.4

a. The bacteria at a time t = 1 hour is obtained by putting t=1 in equation (8):

$N = 25e^{1.386(1)}$, *use a calculator*

$= 99.97$ *bacteria*

b The bacteria at a time t=3 hours is obtained by putting t=3 in equation (8):

$N = 25e^{1.386(3)}$, *use a calculator*

$= 1599$ *bacteria*

c. The bacteria at a time t=4 hours is obtained by putting t=4 in equation (8):

$N = 25e^{1.386(4)}$, *use a calculator*

$= 6392$ *bacteria*

d. Thus, we conclude that the population of bacteria is growing when time t increases.

Logarithmic Functions: Logarithms are an alternative way of writing expressions which involve powers or indices. They are used extensively in the study of sound. The decibel, used in defining the intensity of sound, is based on a logarithmic scale.

Until the development of computers and calculators, logarithms were the only effective tool for large scale numerical computations. They are no longer needed for this, but it still plays a crucial role in many applications.

For illustration, if we start with the exponential function $y=f(x)$ defined by

$y = 2^x$

then the interchange of the variables is giving the inverse of $y = 2^x$:

$x = 2^y$

We call this inverse exponential function, the logarithmic function with base 2, and write this as:

$y = \log_2 x$ if and only if $x = 2^y$ (9)

Definition 1.4.2: [Logarithmic Functions]: The inverse of an exponential function is called a logarithmic function. For b>0 and $b \neq 1$, the logarithmic function is:

$y = \log_b x$, which is equivalent to $x = b^y$

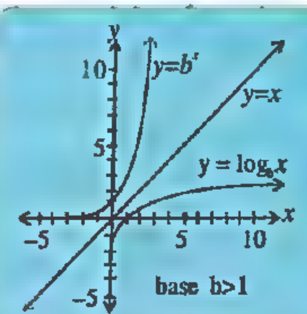


Fig. 1.5

The log to the base b of x is the exponent to which b must be raised to obtain x. The domain of the logarithmic function is the set of all positive real numbers, which is also the range of the corresponding exponential function. Obviously, the range of the logarithmic function is the set of all real numbers, which is also the domain of the corresponding exponential function.

Typical graphs of an exponential function and its inverse, a logarithmic function are shown in the figure (1.5).

Common Logarithms: Common Logarithms (also called Briggsian logarithms) are logarithms with base 10:

Logarithm to the base 10: $y = \log_{10} x$ means $10^y = x$

"Log x", which is read "the logarithm of x", is the answer to the question "to what exponent must 10 be raised to produce x ?

Example 1.4.2: [Logarithmic Function]: Evaluate the following logarithmic functions:.

a. log 10000 b. log .01 c. $\log \sqrt{10} = 1/2$

Solution:

a. This is equal to:

$$\log 10000 = \log 10^4 = 4 \log 10 = 4$$

b. This is equal to:

$$\log .01 = \log \frac{1}{100} = \log(10)^{-2} = -2 \log 10 = -2$$

c. This is equal to:

$$\log\sqrt{10} = \log(10)^{1/2} = \frac{1}{2}\log 10 = \frac{1}{2}$$

Natural Logarithms: Natural Logarithms (also called Napierian logarithms) are logarithms with base e:

Logarithm to the Natural Base e: $y = \ln x$ means $e^y = x$

"ln x" is the answer to the question" to what exponent must e be raised to produce x" ?

Logarithm to the Base b : $y = \log_b x$ means $b^y = x$

" $y = \log_b x$ ", which is read "y is the logarithm of x to the base b", is the answer to the question " to what power must b be raised to produce x?

Logarithmic Notation:

Common logarithmic:	$\log x = \log_{10} x$
Natural logarithmic:	$\ln x = \log_e x$

Logarithmic-Exponential Relationships:

$\log x = y$	**is equivalent to**	$x = 10^y$
$\ln x = y$	is equivalent to	$x = e^y$

Properties of Logarithms:

If b, M and N are positive real numbers, $b \neq 1$, and p and x are also any positive real numbers, then :

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = x$
4. $b^{\log_b x} = x, x > 0$
5. $\log_b MN = \log_b M + \log_b N$
6. $\log_b \frac{M}{N} = \log_b M - \log_b N$
7. $\log_b M^p = p\log_b M$
8. $\log_b M = \log_b N, M = N$

Example 1.4.3:[Logarithmic Functions]: Find x to four decimal places for the following indicated exponential functions:

a. $10^x = 2$, b. $e^x = 3$, c. $3^x = 4$

Solution:

$$10^x = 2$$

a.
$$\log 10^x = \log 2, \quad \log \textit{ of both sides}$$
$$x \log 10 = \log 2, \quad \log 10 = 1$$
$$x = 0.3010, \quad \textit{Calculator value}$$

$$e^x = 3$$

b.
$$\ln e^x = \ln 3, \quad \ln \textit{ of both sides}$$
$$x \ln x = \ln 3, \quad \ln x = 1$$
$$x = 1.0986, \quad \textit{Calculator value}$$

$$3^x = 4$$

$$\log 3^x = \log 4, \quad \log \textit{ of both sides}$$

c.
$$x \log 3 = \log 4,$$
$$x = \frac{\log 4}{\log 3} = 1.2619, \quad \textit{Calculator value}$$

Example 1.4.4:[Logarithmic Function]: Find x so that

$\frac{3}{2}\log_b 4 - \frac{2}{3}\log_b 8 + \log_b 2 = \log_b x.$

Solution:

$$\frac{3}{2}\log_b 4 - \frac{2}{3}\log_b 8 + \log_b 2 = \log_b x$$
$$\log_b 4^{\frac{3}{2}} - \log_b 8^{\frac{2}{3}} + \log_b 2 = \log_b x,$$
$$\log_b 8 - \log_b 4 + \log_b 2 = \log_b x$$
$$\log_b \frac{(8)(2)}{4} = \log_b x,$$
$$\log_b 4 = \log_b x$$
$$x = 4$$

Hyperbolic functions: In physics, it is shown that a heavy, flexible cable (for example a power line) that is suspended between two points at the same height assumes the shape of a curve called a catenary, with an equation of the form

$$y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right) \qquad (10)$$

This is one of several important applications that involve combinations of exponential functions. In certain ways, the functions we shall study are analogous to the trigonometric functions, and they have essentially the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason, these functions are called hyperbolic functions. Three basic functions are the hyperbolic sine (denoted "sinhx" and pronounced "cinch"), the hyperbolic cosine (coshx; pronounced "kosh") and the hyperbolic tangent (tanhx; pronounced "tansh"). They are listed as under:

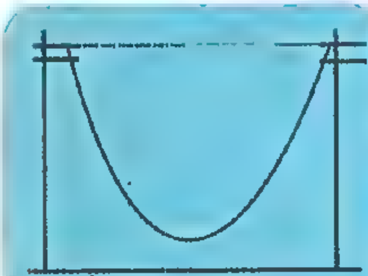


Fig. 1.6

$$\sinh x=\frac{e^x-e^{-x}}{2},\quad \cosh x=\frac{e^x+e^{-x}}{2},\quad \tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}},\quad \coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}}$$

$$\cosh^2 x-\sinh^2 x=1$$

$$\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$$

$$\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y$$

The name "hyperbolic functions" comes from the fact that the functions sinht and cosht play the same role in the parametric representation of the hyperbolic

$$x^2-y^2=1,$$

as the trigonometric functions sint and cost, do in the parametric representation of the circle

$$x^2+y^2=1.$$

Eliminating the parameter t from the parametric equations

x=cost, y=sint

to obtain the equation of the circle:

$$x^2+y^2=\cos^2 t+\sin^2 t=1$$

Similarly, the equations

x=cosht, y=sinht

are the parametric equations of the hyperbola. Squaring these equations and subtracting the second from the first to obtain the equation of hyperbola:

$$x^2-y^2=\cosh^2 t-\sinh^2 t=1$$

Explicit and Implicit Functions: So far we have met many functions of the form y=f(x):

$$y = x^2 + 3,\ y = \sin x,\ y = e^{3x} - 2x \qquad (11)$$

If y is equated to an expression involving only x terms, then we say that y is expressed **explicitly** in terms of x (that is in equation 11).

Sometimes we have an equation connecting x and y but it is impossible to write it in the form of y=f(x):

$$y = x^2 - y^3 + \sin x - \cos y = 1,\ \ \sin(x+y) + e^x + e^{-y} = x^3 + y^3 \qquad (12)$$

In these cases we say that y is expressed **implicitly** in terms of x.

Example 1.4.5: [Implicit Equations]: The following curves are modeled through implicit functions:

a. **Ellipse curve:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

b. **Hyperbola curve:** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Parametric Representation of Curves: It is sometimes useful to define the variables x and y in the ordered pair (x, y), so that they are each functions of some other variable, say t:

$$x = f(t) \quad \text{and} \quad y = g(t) \qquad (13)$$

The domain of these functions f(t) and g(t) is some interval D. The variable t is called a parameter and $x=f(t)$ and $y=g(t)$ are called the parametric equations.

Definition 1.4.3: [Parametric Equations]: If f(t) and g(t) are continuous functions of parameter t on an interval D, then the equations

$$x=f(t) \quad \text{and} \quad y=g(t)$$

are called the parametric equations for the plane curve generated by the set of ordered pairs in the plane:

$$(x, y)=(x(t), y(t))=(f(t), g(t)). \qquad (14)$$

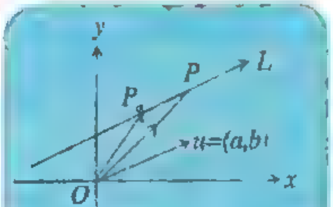


Fig. 1.7

Example 1.4.6:[Parametric Equations of a Line]: Draw the graph of the parametric equations of a line through point $P_0(x_0, y_0)$ and parallel to a direction vector u=(a, b).

Solution: The parametric equations of a line through $P_0(x_0, y_0)$ parallel to u=(a, b) are:

$$x = x_0 + at,\ \ y = y_0 + bt$$

The graph developed is shown in the figure (1.7). For graphical details, see Example 1.5.6

Exercise 1.2

1. Simplify the following functions:

   a. $(4^{3x})^{2y}$ b. $10^{3x-1}10^{4-x}$ c. $\dfrac{e^{x-1}}{e^{x-4}}$ d. $\dfrac{e^{x}}{e^{3-x}}$

   e. $\left(2e^{1.2t}\right)^3$ f. $\left(3e^{-1.4x}\right)^2$

2. Solve the following equations:

   a. $10^{2-3x}=10^{5x-6}$ b. $5^{3x}=5^{4x-2}$ c. $4^{5x-x^2}=4^{-6}$

   d. $7^{x^2}=7^{2x+3}$ e. $5^3=(x+2)^3$ f. $(1-x)^5=(2x-1)^5$

   g. $(x-3)e^x=0$ h. $2xe^{-x}=0$ i. $3xe^{-x}+x^2e^{-x}=0$

   j. $x^2e^x-5xe^x=0$

3. Rewrite in equivalent exponential form, the following logarithmic functions:

   a. $\log_3 27=3$ b. $\log_2 32=5$ c. $\log_{10} 1=0$

   d. $\log_e 1=0$ e. $\log_4 8=\dfrac{3}{2}$ f. $\log_9 27=\dfrac{3}{2}$

4. Rewrite in equivalent logarithmic form, the following exponential functions:

   a. $49=7^2$ b. $36=6^2$ c. $8=4^{3/2}$

   d. $9=27^{2/3}$ e. $A=b^u$ f. $M=b^x$

5. Find x, y and b without a scientific calculator use:

   a. $\log_3 x=2$ b. $\log_2 x=2$ c. $\log_7 49=y$

   d. $\log_b 10^{-4}=-4$ e. $\log_{1/3} 9=y$ f. $\log_b 1{,}000=\dfrac{3}{2}$

6. Solve the following equations for the unknown x:

   a. $\log_b x=\dfrac{2}{3}\log_b 8+\dfrac{1}{2}\log_b 9-\log_b 6$

   b. $\log_b x=\dfrac{2}{3}\log_b 27+2\log_b 2-\log_b 3$

   c. $\log_b x=\dfrac{3}{2}\log_b 4-\dfrac{2}{3}\log_b 8+2\log_b 2$

   d. $\log_b x+\log_b(x-4)=\log_b 21$

e. $\log_{10}(x-1)-\log_{10}(x+1)=1$

f. $\log_{10}(x+6)-\log_{10}(x-3)=1$

7. Suppose the sales of a certain product are approximated by

$S(t)=125+83\log(5t+1)$,

where S(t) is sales in thousands of dollars t years after the product was introduced on the market. Find

a. S(0) b. S(2) c. S(4) d. S(31)

e. Draw and interpret the graph.

1.5 Graphical Representation of Functions

i) *Graphical views of the given functions*

The views of some particular functions are graphed in the following examples:

Example 1.5.1:[Graph of 2^x]: Sketch a graph of $y=2^x$.

Solution: Many students, if asked to hand sketch graphs of equations such as $y=2^x$ or $y=2^{-x}$, would not hesitate at all. They would likely make up tables by assigning integers to x, plot the resulting points, and then join these points with a smooth curve as in figure (1.8). The only sketch is that we have not defined 2^x for all real numbers. We know that 2^5, 2^{-3}, $2^{2/3}$, $2^{-3/5}$, $2^{1.4}$ and $2^{-3.14}$ mean (that is, 2^p, where p is a rational number), but what does $2^{\sqrt{2}}$ mean? The question is not easy to answer at this time. In fact, a precise definition of $2^{\sqrt{2}}$ must wait for more advanced courses, where it is shown that 2^x names a positive real number for x any real number, and that the graph of $y=2^x$ is indicated in figure(1.8).

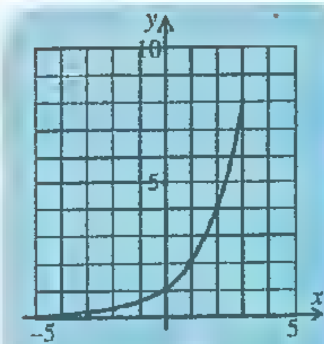


Fig. 1.8

It is useful to compare the graphs $y=2^x$ and $y=2^{-x}$ by plotting both on the same set of coordinate axes. These are shown in figures 1.9(a) and 1.9(b). The graph of

$f(x)=b^x,\ b>1$ figure 1.9(a)

looks very much like the graph of $y=2^x$, and the graph of

$f(x)=b^x,\ 0<b<1$ figure 1.9 (b)

looks very much like the graph of $y = 2^{-x}$.

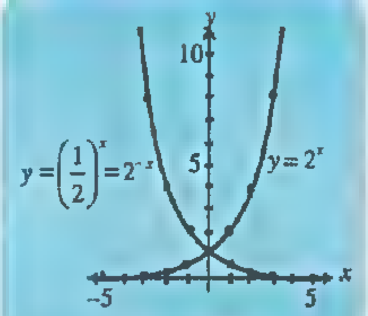


Fig. 1.9 (a)

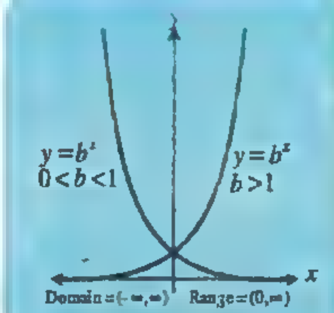


Fig. 1.9 (b)

The graphs in figures 1.9(a) and 1.9(b) suggest the following important general properties of exponential functions that are summarized in the box:

Basic properties of the graph of $f(x) = b^x, b > 0, b \neq 1$

1. All graphs will pass through the point (0,1).
2. All graphs are continuous curves, with no holes or jumps.
3. If $b > 1$, then b^x increases as x increases.
4. If $0 < b < 1$, then b^x decreases as x increases.

Example 1.5.2:[Exponential Function]: Sketch a graph of $y = \left(\frac{1}{2}\right)4^x, -2 \le x \le 2.$

Solution: Use a scientific calculator to create the table of points. Plot these points and then join them to obtain the graph of a smooth curve in the figure 1.10.

x	-2	-1	0	1	2
y	0.031	0.125	0.50	2.00	8.00

Example 1.5.3: [Graph of e^x and e^{-x}]: Draw the graphs of $y = e^x$ and $y = e^{-x}$.

Solution: Use a scientific calculator to create the table of points. Plot these points and then join them to obtain the graphs of smooth curves in the figure (1.11). The domain is $(-\infty,\infty)$, while the range is $(0,\infty)$.

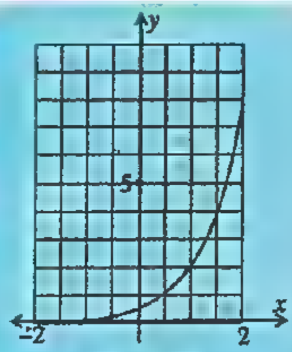


Fig. 1.10

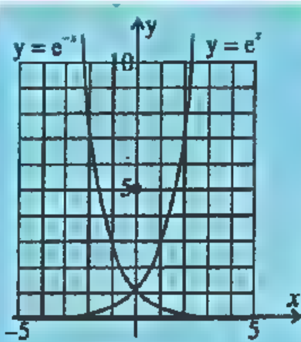


Fig. 1.11

Example 1.5.4: [Graph of $\log_2 x$]: Sketch the graph of $y = \log_2 x$.

Solution: We can graph $y = \log_2 x$ by plotting $x = 2^y$, since they are equivalent. Any ordered pair of numbers on the graph of the exponential function will be on the graph of the logarithmic function if we interchange the order of the components. For example, ordered pair (3,8) satisfies $y = 2^x$ and (8,3) satisfies equation $x = 2^y$. The graphs of $y = 2^x$ and $y = \log_2 x$ are shown in the figure (1.12):

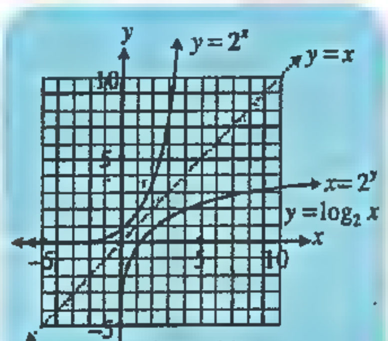


Fig. 1.12

Example 1.5.5: [Graph of $y = \ln x$]: Sketch the graph of $y = \ln x$.

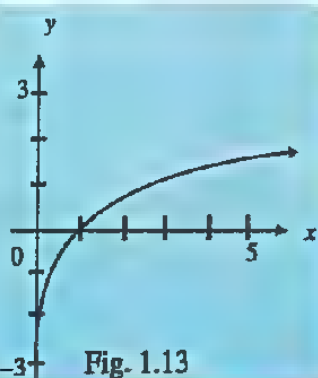


Fig. 1.13

Solution: We can graph $y = \ln x$ by plotting $x = e^y$, since they are equivalent. Any ordered pair of numbers on the graph of the exponential function will be on the graph

of the logarithmic function if we interchange the order of the components. The graph is shown in the figure (1.13).

Example 1.5.6: [Graph of Parametric Equation]: Sketch the graph of the parametric functions $(x(t), y(t)) = (3 - t, 2t)$ for all t

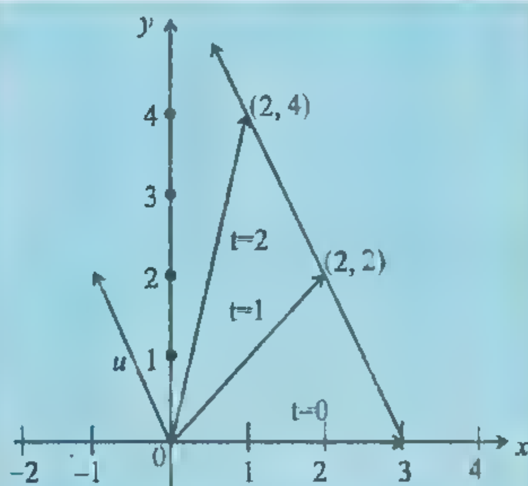


Fig. 1.14

Solution: The graph is the collection of all points (x, y) with $x=3-t$, $y=2t$ for different real values of t:

$t=0 \Rightarrow (x(t),y(t))=(3,0)$

$t=1 \Rightarrow (x(t),y(t))=(2,2)$

$t=2 \Rightarrow (x(t),y(t))=(1,4)$

The plot of the position vectors $t_0=(3,0), t_1=(2,2), t_2=(1,4)$ in the figure (1.14) developed a straight line parallel to the direction vector u=(-1,2) and passing through the point p(3,0).

Example 1.5.7:[Compound Function]: Graph the compound function:

$$f(x)=\begin{cases} 3-x & if \quad x<-2 \\ x+2 & if \quad -2\le x<2 \\ 1 & if \quad x\ge 2 \end{cases}$$

Solution:

a. Use the function $f(x)=3-x$ for $x<-2$ to obtain a set of points:

x:	-3	-2
$f(x)$:	6	5

b. Use the function $f(x)=x+2$ for $-2 \le x < 2$ to obtain a set of points:

x:	-2	2
$f(x)$:	0	4

c. Use the function $f(x)=1$ for $x \ge 2$ to obtain a set of points:

x:	2	4
$f(x)$:	1	1

Use these tabular points to obtain the graph of a compound function:

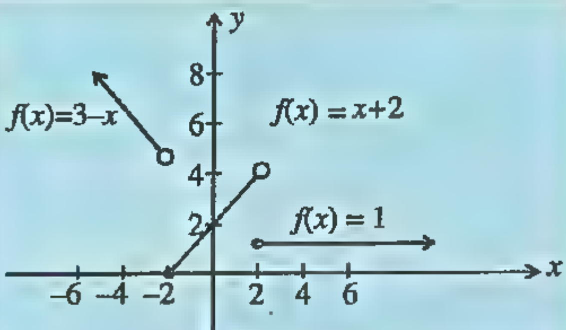


Fig. 1.15

This function is also called a **piecewise continuous function** which is discontinuous (piecewise continuous) at $x=-2$ and $x=2$.

Exercise 1.3

1. Using a calculator and point–by–point to plot the following exponential functions:

a. $h(x)=x(2^{x});[-5,0]$ b. $m(x)=x(3^{-x});[0,3]$

c. $N=\dfrac{100}{1+e^{-t}};[0,5]$ d. $N=\dfrac{200}{1+3e^{-t}};[0,5]$

2. Using a calculator and point–by–point to plot the following logarithmic functions:

a. $y=\ln x$ b. $u=-\ln x$ c. $y=2\ln(x+2)$

d. $y=4\ln(x-3)$ e. $y=4\ln x-3$

3. Sketch the following parametric curves:

a. $(x(t),y(t))=(3-t,2t)$, t is real number.

b. $(x(t),y(t))=(4\cos t,3\sin t)$

c. $(x(t), y(t), z(t)) = (3+2t, 5-3t, 2-4t)$

1.6 Limit of a Function

The algebraic problems considered in earlier sections dealt with static situations:

What is the revenue when x items are sold?

How much interest is earned in 2 years?

Calculus, on the other hand, deals with dynamic situations:

At what rate is the economy growing?

How fast is a rocket going at any instant after lift–off?

The techniques of calculus will allow us to answer many questions like these that deal with rates of change.

The key idea underlying the development of calculus is the concept of limit. So we begin by studying limits after explaining the location of intervals on the real number line.

i) *Identification of real number by a point on the number line*

The various types of numbers used in this book can be illustrated with a diagram called a number line. Each real number corresponds to exactly one point on the line and vice–versa. A number line with several sample numbers located on it is shown in figure (1.16):

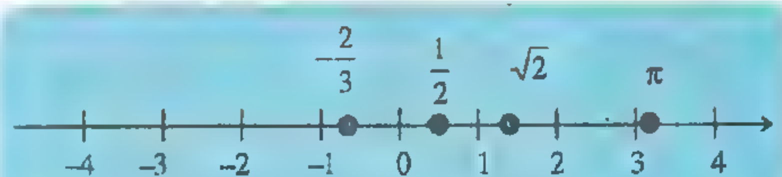


Fig 1.16

Example 1.6.1:[Real Numbers]: What kind of number is each of the following:

a. 6? b. ¾? c. $\sqrt{8}$?

Solution

a. The number 6 is a natural number, whole number, integer, rational and real number.

b. The number 3/4 is rational and real.

c. The number $\sqrt{8}$ is irrational and real.

Example 1.6.2: [Real Numbers]: Use a calculator to approximate the following numbers to three significant digits:

a. $\sqrt{850}$ b. $-4/7$ c. $\pi/3$

Solution:

a. The approximation to three significant figures of a given number is 29.2.

b. The approximation to three significant figures of a given number is $-.571$

c. The approximation to three significant figures of a given number is 1.05.

The comparison of two real numbers requires symbols that indicate their order on the number line. The following symbols are used to indicate that one number is greater than or less than another number:

$<$ means is less than	$\leq$ means is less than or equal to
$>$ means is greater than	$\geq$ means is greater than or equal to

The following definitions show how the number line is used to decide which of two given numbers is the greater:

For real numbers a and b,

if a is to the left of b on a number line, then $a<b$.

if a is to the right of b on a number line, then $a>b$.

Example: 1.6.3:[Real Number Inequality]: Write true or false for each of the following inequalities:

a. $8<12$ b. $-6>-3$

Solution:

a. $8<12$

This real inequality says that 8 is less than 12, which is true. The graph on a real number line is:

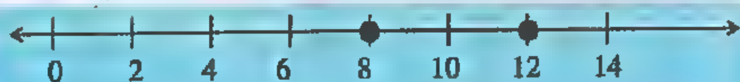


Fig. 1.17

b. $-6>-3$

The real inequality $-6<-3$ says that -6 is less than -3 (-6 is to the left of -3). The given inequality $-6>-3$ is false. The graph of $-6<-3$ on the real number line is:

Fig. 1.18

Example 1.6.4:[Real Number Line]: Graph all integers x on the real number line such that $1<x<5$.

Solution: The only integers between 1 and 5 are 2, 3 and 4. These integers are graphed on the number line by solid holes in the figure below

Fig. 1.19

Example 1.6.5:[Real Number Line]: Graph all real numbers x on the real number line such that $1<x<5$.

Solution: The graph includes all the real numbers between 1 and 5 and not just the integers. Graph these numbers by drawing a heavy line from 1 to 5 on the number line, as in figure below. Open holes at 1 and 5 show that neither of these points belongs to the graph.

Fig. 1.20

ii) Definition of closed, open, half open and half closed intervals

A set that consists of all the real numbers between two points is called an interval. A special notation will be used to indicate an interval on the real number line.

For example, the interval including all numbers x, where $-2<x<3$ is written as $(-2,3)$. The parentheses indicate that the number -2 and 3 are not included. If -2 and 3 are to be included in the interval, square brackets are used, as in $[-2,3]$. The chart below shows several typical intervals, where $a<b$:

Inequality	Interval Notation	Explanation
• $a \le x \le b$	[a,b]: Closed	Both a and b are included.
• $a \le x < b$	[a,b): Half open/Closed	a is included, b is not.
• $a < x \le b$	(a,b]: Half Open/Closed	b is included, a is not.
• $a < x < b$	(a,b): Open	Both a and b are not included.

Interval notation is also used to describe sets such as the set of all numbers x, with $x \ge -2$. This interval is written $[-2,\infty)$.

Example 1.6.6:[Interval]: Graph the following intervals:

a. $[-2,\infty)$ b. $[4,\infty)$ c. $[-2,1]$

Solution:

a. Start at -2 and draw a heavy line to the right, as in graph. Use a solid hole at -2 to show that -2 is itself a part of the graph. The symbol ∞, read "infinity" does not represent a number. It simply indicates that all numbers greater than -2 are in the interval. Similarly, the notation $(-\infty, 2)$ indicates the set of all real numbers with $x < 2$.

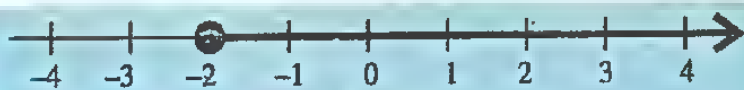


Fig. 1.21

b. The graph of the interval $[4,\infty)$ is as under:

Fig. 1.22

c. The graph of the interval $[-2,1]$ is as under:

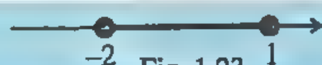


Fig. 1.23

Example 1.6.7:[Interval]: Use each of the graph to indicate the interval notation:

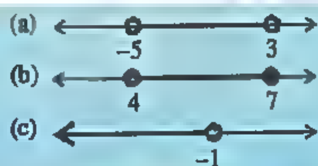


Fig. 1.24

Solution: The given graphs indicate the following intervals:

a. (-5, 3) b. [4,7] c. $(-\infty,-1]$

iii) Explanation of phrases $x \to 0,\ x \to a,\ x \to \infty$

The answer to the phrase x tends to "0" is easy to see that the value of a function

$$y = f(x) = \frac{x^2 - 4}{x - 2}$$

gets closer and closer to a single real number "2"on both left and right sides of "2", when x is a number very close to "0" on both left and right sides of "0". In this situation, we are in position to say that x approaches to "0" or x tends to "0" and is denoted by $x \to 0$, when $f(x)$ tends to a single number "say L=2".

The answer to the phrase x tends to "infinity" is easy to see that the function

$$f(x)=\frac{3x+2}{x+1}$$

gets smaller and smaller, when x approaches "infinity" from either side of a number say 3. In this situation, we say that the function f(x) gets closer and closer to a single number 3 when $x \to \infty$ from either side.

iv) Definition of the limit of a function

To develop the concept of a limit, suppose the concentration of a drug in a patient's bloodstream h hours after it was injected is given by:

$$A(h)=\frac{0.2\,h}{h^2+2}$$

The questions to ask are the following:

- what is the drug concentration after h=0.5 hour?
- what is the drug concentration after h=1 hour?
- what is the limit of drug concentration at h=1 hour in a patient's bloodstream, that is $\lim\limits_{h\to 1} A(h)$?

The answer to drug concentration in a patient's bloodstream after h=0.5 hour is obtained by putting h=0.5 in a function A(h):

$$A(0.5)=\frac{(0.2)(0.5)}{(0.5)^2+2}=0.0444$$

The answer to drug concentration in a patient's bloodstream after h=1 hour is obtained by putting h=1 in a function A(h):

$$A(1)=\frac{(0.2)(1)}{(1)^2+2}=0.06667$$

The answer to the limit of a drug concentration in a patient's bloodstream at h=1 hour is easy to see that the value of a rational function

$$A(h)=\frac{0.2\,h}{h^2+2}$$

gets closer and closer to a single real number 0.06667 (on both left and right sides of 0.06667), when h is a number very close to 1 hour (on both left and right sides of 1, that is h<1 and h>1))

h<1(left)	A(h)	h>1(right)	A(h)
0.99	0.06644	1.02	0.06710
0.999	0.066644	1.01	0.066885
0.9999	0.0666644	1.001	0.0666887
0.99999	0.66666444	1.0001	0.06666689

This situation is usually described by saying that " the limit of a function A(h) as h approaches 1 is the number 0.6667", which is written symbolically as:

$$\lim_{h\to 1} A(h)=\lim_{h\to 1}\frac{0.2h}{h^2+2}=0.6667$$

Definition 1.6.1: [Limit of a Function]: If $f(x)$ is a function of x, and c, L are the real numbers, then L is the limit of a function $f(x)$ as x approaches c:

$$\lim_{x\to c} f(x)=L$$

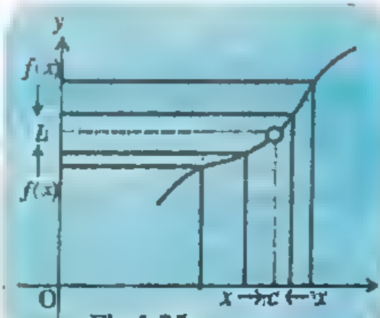


Fig.1.25

The graph clears that as x gets closer and closer to c (on both sides of c), the corresponding value of $f(x)$ gets closer and closer (possibly are equal) to L.

The illustration and the definition indicate that a limit exists only,

- when $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side.
- when the limiting value of a function from the left equals the limiting value of a function from the right.

Example 1.6.8: Show how $f(x)=\dfrac{x^2-4}{x-2}$ behaves as x approaches 2?

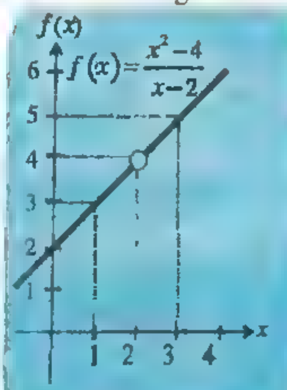


Fig. 1.26

Solution: The graph of a given function is shown in the figure (1.26).

What happens to the denominator of the rational function, when x approaches 2? As the values of x gets closer and closer to 2, the denominator approaches zero,

and division by zero is undefined. When $x = 2$, $f(2) = -4/0$, which is undefined. This means that the function is not defined at $x=2$ which is indicated by a symbol "open hole" in the graphical view of a rational function.

However, we are asked to show how the function behaves as x gets closer and closer to 2. To do this, the tabular form of a rational function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

as x approaches 2 from the left and the right,

$x<2$	f(x)	$x>2$	f(x)
1	3	3	5
1.5	3.5	2.5	4.5
1.9	3.9	2.1	4.1
1.99	3.99	2.01	4.01
1.999	3.999	2.001	4.001
1.9999	3.9999	2.0001	4.0001
1.99999	3.99999	2.00001	4.00001
1.999999	3.999999	2.000001	4.0000001

the function $f(x)$ gets closer and closer to 4. Thus, the limit of a function, as x approaches 2, is 4. Symbolically, this leads the notation:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)} = \lim_{x\to 2}(x+2) = 4$$

v) *Theorems on limits of functions*

Theorem 1.1:[Limit Theorem of a Function]: If $f(x)$ and $g(x)$ are two functions of x, and the limits of these two functions are $\lim_{x\to c} f(x) = A$, $\lim_{x\to c} g(x) = B$. when exist, then the limit-rules developed are the following:

1. **Constant Rule:** If $f(x) = k$, k is any constant, then
$$\lim_{x\to c} f(x) = \lim_{x\to c} k = k$$

2. **Limit Rule:** $\lim_{x\to c} x = c$

3. **Multiple Rule:** $\lim_{x\to c} k\ f(x) = k \lim_{x\to c} f(x)$

The limit of constant times a function is the constant times the limit of a function.

4. **Sum/Difference Rule:** $\lim_{x\to c}[f(x)\pm g(x)]=\lim_{x\to c} f(x)\pm\lim_{x\to c} g(x)$

The limit of a sum or difference is the sum or difference of the limits.

5. **Polynomial Rule:** If p(x) is a polynomial, then

$$\lim_{x\to c} p(x)=p(c).$$

6. **Product Rule:** $\lim_{x\to c}[f(x)\ g(x)]=\lim_{x\to c} f(x)\cdot\lim_{x\to c} g(x)$

The limit of a product is the product of the limits.

7. **Quotient Rule:** $\lim_{x\to c}\dfrac{f(x)}{g(x)}=\dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)},\ \lim_{x\to c} g(x)\neq 0$

The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero.

8. **Power Rule:** For any rational number n,

$$\lim_{x\to c}[f(x)]^n=[\lim_{x\to c} f(x)]^n$$

The limit of a power is the power of the limit.

9. **Equal Functions Rule:** If $f(x)=g(x)$ for all x, then

$$\lim_{x\to c} f(x)=\lim_{x\to c} g(x)$$

vi) *Definition of the limit of a sequence*

In previous sections, most phenomena we have considered occur continuously, but practically in every field of inquiry, there are situations that can be described by cataloging individual items on a numerical listing. For instance, a drug is administrated into the body. At the end of each hour, the amount of drug present is half what it was at the end of the previous hour. A mathematician might refer to the injecting drugs labeling procedure by saying the amounts of drugs are arranged in a **sequence.**

A sequence is a succession of numbers that are listed according to a given prescription or rule. Specifically, if n is a positive integer, then the sequence whose **nth term** is the number s_n can be written as

$s_1, s_2, s_3, \ldots, s_n, \ldots$ or more simply $\{s_n\}$

The number s_n is called the **general term** (or **nth term**) of the sequence. We will deal only with infinite sequences, so each term s_n has a **successor** s_{n+1} and for n>1, a **predecessor** s_{n-1}. For example, the sequence of each positive number with its reciprocal $\frac{1}{n}$ is denoted by $\left\{\frac{1}{n}\right\}$ which represents the succession of numbers 1, 1/2, 1/3,. ...,1/n ...whose general term is $s_n = \frac{1}{n}$.

So far, we have been discussing the concept of a sequence informally, without definition. We have been observed that a sequence $\{s_n\}$ associates the number s_n with the positive integer n. Hence, a sequence is really a special kind of function, one whose domain is the set of all positive (or possibly, nonnegative) integers.

Definition 1.6.2: [Definition of a Sequence]: A sequence $\{s_n\}$ is a function whose **domain** is a set of all nonnegative integers and whose **range** is a subset of the real numbers. The functional values $s_1, s_2, s_3, \ldots$ that are the terms of the sequence $\{s_n\}$ now takes the **functional notation**

$s(n),\ n = 1, 2, 3, \ldots$

Example 1.6.9: [Graph of a Sequence]: Graph the sequence on one as well as on two dimensional space whose nth term is $s_n = \frac{8n}{n+3}$.

Solution: The sequence $\{s_n\}$ in terms of function notation is $s(n) = \frac{8n}{n+3}$, whose domain is the set of nonnegative integers. The functional values of s(n) develop

n: Integers	Function: s(n)
1	$s_1 = s(1) = 2.0$
2	$s_2 = s(2) = 3.2$
3	$s_3 = s(3) = 4.0$
4	$s_4 = s(4) = 4.57$

a. the one dimensional view on a real number line:

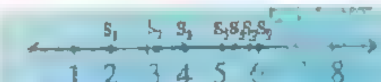


Fig. 1.27

b. the two dimensional view :

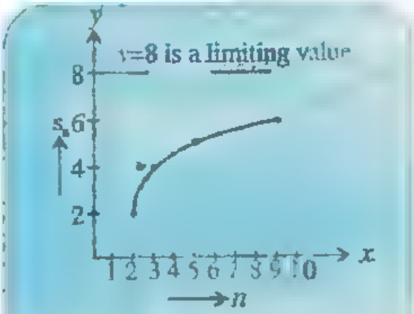


Fig. 1.28

It is desirable to examine the behavior of a given sequence $\{s_n\}$ as n gets arbitrary large. For example, we can plot the terms of the given sequence $\{s_n\}=\left\{\frac{8n}{n+3}\right\}$ on a number line in Fig. 1.27. The sequence is also viewed on a two dimensional space in Fig. 1.28. By looking at either graph, we see that it appears the terms of the sequence are approaching 8. In general, if the terms of the sequence approach the number L as n increases without bound, we say that the sequence **converges** to the limit L and write

$$\lim_{n\to\infty} s_n = L$$

For instance, in our example, we would expect

$$\lim_{n\to\infty} s_n = \lim_{x\to\infty}\frac{8n}{n+3} = 8 \qquad (14)$$

Definition 1.6.3: [Limit of a Sequence]: A sequence $\{s_n\}$ is converging to a real number L:

$$\lim_{n\to\infty} s_n = L \qquad (15)$$

If for every h>0, there is an integer N, such that

$$|s_n - L| < h \qquad \text{whenever} \qquad n \geq N \qquad (16)$$

Otherwise, the sequence **diverges.**

vii) Determination of a limit of a sequence, when the N-th term is known

The N-th term of a sequence $\{s_n\}=\left\{\frac{8n}{n+3},\ n=1,2,3,...\right\}$ is $\frac{8n}{n+3}$. How to show that the sequence is converging to a number 8?

In this situation, the convergence or divergence of a sequence can be found by the rules developed in the following theorem 1.2.

Theorem 1.2:[Limit Theorem of Sequence]: If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then the limit exist are the following:

1. **Linearity rule:** $\lim_{n\to\infty}(ra_n + sb_n) = rL + sM$
2. **Product rule:** $\lim_{n\to\infty}(a_n b_n) = LM$
3. **Quotient rule:** $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{L}{M},\ M \neq 0$
4. **Root rule:** $\lim_{n\to\infty}\sqrt[m]{a_n} = \sqrt[m]{L}$

Example 1.6.10: Find the limit of each of these convergent/divergent sequences:

a. $\left\{\frac{8n}{n+3}\right\}$ b. $\left\{\frac{n^5+n^3+2}{7n^4+n^2+3}\right\}$ c. $\left\{(-1)^n\right\}$

Solution:

a. Divide the numerator and denominator by n to obtain the required limit:

$$\lim_{n\to\infty}\left\{\frac{8n}{n+3}\right\} = \lim_{n\to\infty}\frac{n(8)}{n\left(1+\frac{3}{n}\right)} = \frac{8}{1} = 8$$, sequence is converging to 8

b. Divide the numerator and denominator by n^5 to obtain:

$$\lim_{n\to\infty}\left\{\frac{n^5+n^3+2}{7n^4+n^2+3}\right\} = \lim_{n\to\infty}\frac{n^5\left[1+\frac{1}{n^2}+\frac{2}{n^5}\right]}{n^5\left[\frac{7}{n}+\frac{1}{n^3}+\frac{3}{n^5}\right]}$$, sequence is diverging

The numerator tends toward 1 as $n \to \infty$, and the denominator approaches 0. Hence the quotient increases without bound and the sequence must diverge.

c. The sequence defined by $\left\{(-1)^n\right\}$ is -1, 1,-1,1,....This sequence diverges by oscillation, because the nth term is always either 1 or -1, so the sequence cannot approach one specific number say L as n grows large.

vii) *Evaluation of limits for different algebraic, exponential and trigonometric functions*

The idea of limits in the above situations is given in the following examples:

Example 1.6.11: Evaluate $\lim\limits_{x\to 1}\dfrac{x^3-3x+7}{5x^2+9x+6}$.

Solution:

$$\lim_{x\to 1}\frac{x^3-3x+7}{5x^2+9x+6}=\frac{\lim\limits_{x\to 1}(x^3-3x+7)}{\lim\limits_{x\to 1}(5x^2+9x+6)}=\frac{1-3+7}{5+9+6}=\frac{5}{20}=\frac{1}{4}$$

Example 1.6.12:[Power Function]: Evaluate $\lim\limits_{x\to -2}\sqrt[3]{(x^2-3x-2)}$.

Solution:

$$\lim_{x\to -2}\sqrt[3]{x^2-3x-2}=\lim_{x\to -2}(x^2-3x-2)^{1/3}$$

$$=[\lim_{x\to -2}(x^2-3x-2)]^{1/3}=[(-2)^2-3(-2)-2]^{1/3}=(8)^{1/3}=2$$

Example 1.6.13: [Trigonometric Function]: Evaluate the limits

a. $\lim\limits_{x\to 0}\sin^2 x$ b. $\lim\limits_{x\to 0}(1-\cos x)$, when $\lim\limits_{x\to 0}\sin x=0$ and $\lim\limits_{x\to 0}\cos x=1$.

Solution:

a. $$\lim_{x\to 0}\sin^2 x=\left[\lim_{x\to 0}\sin x\right]^2=0$$

b. $$\lim_{x\to 0}(1-\cos x)=\lim_{x\to 0}1-\lim_{x\to 0}\cos x=1-1=0$$

Example 1.6.14: [Fraction Reduction]: Evaluate $\lim\limits_{x\to 2}\dfrac{x^2+x-6}{x-2}$.

Solution: If you are trying to substitute direct this limit, then you will obtain:

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}=\frac{4+2-6}{2-2}=\frac{0}{0}$$

The form 0/0 is called indeterminate form. In other words, we cannot evaluate a limit for which direct substitution yields 0/0.

If the expression is a rational expression, then the next step is to factorize the function after simplification to see if the reduced form is a polynomial.

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}=\lim_{x\to 2}\frac{(x+3)(x-2)}{(x-2)}=\lim_{x\to 2}(x+3)=2+3=5$$

Example 1.6.15: [Function Rationalization]: Evaluate $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$.

Solution: Once again, notice that both the numerator and denominator of this rational expression are 0 when $x=4$. We cannot evaluate the limit by direct substitution. Instead, we need to multiply and divide out the expression by the conjugate of the numerator:

$$\begin{aligned}\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}&=\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}\\&=\lim_{x\to 4}\frac{x-4}{(x-4)(\sqrt{x}+2)}\\&=\lim_{x\to 4}\frac{1}{\sqrt{x}+2}\\&=\frac{1}{\sqrt{4}+2}=\frac{1}{4}\end{aligned}$$

Example 1.6.16: [Trigonometric Function]:

Solution: Both the numerator and denominator of this quotient are going to be 0, when $x=0$. We need to multiply both the numerator and denominator of $f(x)$ by $1+\cos x$ (conjugate) to obtain:

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x}&=\lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\\&=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\&=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{1+\cos x}\right)\\&=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\lim_{x\to 0}\left(\frac{\sin x}{1+\cos x}\right)=1\left(\frac{0}{1+1}\right)=1(0)=0\end{aligned}$$

Important Limits

i) Evaluation of the limits of the following functions

1. $\lim\limits_{x\to a}\left(\dfrac{x-a}{\sqrt{x}-\sqrt{a}}\right)=2\sqrt{a}$

When $x\to a$, the limit of a function is of the form (0/0), which is indeterminte. In this situation, we need to rationalize the given function to obtain the required limit:

$$\begin{aligned}\lim_{x\to a}\left(\frac{x-a}{\sqrt{x}-\sqrt{a}}\right)&=\lim_{x\to a}\left(\frac{x-a}{\sqrt{x}-\sqrt{a}}\right)\times\left(\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right)\\&=\lim_{x\to a}\frac{(x-a)(\sqrt{x}+\sqrt{a})}{x-a}\\&=\lim_{x\to a}(\sqrt{x}+\sqrt{a})\\&=\lim_{x\to a}\sqrt{x}+\lim_{x\to a}\sqrt{a}\\&=\sqrt{a}+\sqrt{a}\\&=2\sqrt{a}\end{aligned}$$

2. $\lim\limits_{x\to a}\left(\dfrac{x^n-a^n}{x-a}\right)=na^{(n-1)}$

Again the limit is undefined.. In this situation, we need to divide out the numerator by denominator to obtain:

$$\frac{x^n-a^n}{x-a}=x^{n-1}+ax^{n-2}+a^2x^{n-3}+........+a^{n-2}x+a^{n-1} \qquad (17)$$

Being a polynomial, the function to the right of the above expression (17) is continuous for all values of x and as such its limit, when $x\to a$ must equal to its value at x=a. Thus, the limit of the expression (17), when x tends to a is:

$$\begin{aligned}\lim_{x\to a}\frac{x^n-a^n}{x-a}&=\lim_{x\to a}(x^{n-1}+ax^{n-2}+a^2x^{n-3}+........+a^{n-2}x+a^{n-1})\\&=a^{n-1}+aa^{n-2}+a^2a^{n-3}+.......+aa^{n-2}+a^{n-1}\\&=a^{n-1}+a^{n-1}+a^{n-1}+........+a^{n-1}+a^{n-1}=na^{n-1}\end{aligned}$$

3. $\lim_{x \to \infty}\left(1+\frac{1}{x}\right)^x = e$

The base "e" is an irrational number (like π), it cannot be represented exactly by any finite decimal fraction. However, e can be approximated as closely as we like by evaluating the expression

$$\left(1+\frac{1}{x}\right)^x \qquad (18)$$

for sufficiently large x. What happens to the value of the expression as x increases without bound? The results are summarized in the following table:

x	$\left(1+\frac{1}{x}\right)^x$
1	2
10	2.59374...
100	2.70481...
1000	2.71692...
10000	2.71814...
100000	2.71827...
1000000	2.71828...

Interestingly, the value of expression (18) is never close to 1, but seems to be approaching a number close to 2.7183. In fact, as x increases without bound, the value of expression (18) approaches an irrational number that we call e. The irrational number e to twelve decimal places is:

$$\lim_{x \to \infty}\left(1+\frac{1}{x}\right)^x = 2.718281828459 = e \qquad (19)$$

From result (21), the new result deduced is:

$$\lim_{x \to -\infty}\left(1+\frac{1}{x}\right)^x = 2.718281828459 = e$$

4. $\lim_{x \to 0}(1+x)^{\frac{1}{x}} = e$

If we put y=1/x, then $y \to \infty$, when $x \to 0$, and the left-hand side of the limit thus gives the right-hand side:

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y} = e$$

5. $\lim_{x\to 0}\dfrac{a^x-1}{x} = \ln a,\ a>0.$

If we put $a^x-1=y$, then x is obtained by taking log of both sides:

$a^x-1=y$

$a^x=1+y$

$\log(a^x)=\log(1+y)$

$x\log a=\log(1+y)$

$x=\dfrac{\log(1+y)}{\log a}$

Use this x in the expression $(a^x-1)/x$ to obtain:

$$\frac{a^x-1}{x}=\frac{y}{\dfrac{\log(1+y)}{\log a}}$$

Taking limit $y\to 0$, when $x\to 0$:

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{y\to 0}\frac{y}{\dfrac{\log(1+y)}{\log a}}$$

$$=\lim_{y\to 0}\frac{1}{\dfrac{1}{y}\log(1+y)(1/\log a)}$$

$$=\lim_{y\to 0}\frac{\log a}{\log(1+y)^{\frac{1}{y}}}$$

$$= \log a \frac{1}{\log \lim_{y\to 0}(1+y)^{\frac{1}{y}}}$$

$$= \log a \frac{1}{\log e}$$

$$= \log_e a = \ln a$$

The new result deduced from this result is:

$$\lim_{x\to 0}\frac{e^x - 1}{x} = \ln e = 1$$

6. $\lim_{x\to 0}\frac{(1+x)^n - 1}{x} = n$

If we put $(1+x)^n - 1 = z$, then:

$$(1+x)^n - 1 = z$$

$$(1+x)^n = (1+z)$$

Taking log of both sides to obtain:

$$\log(1+x)^n = \log(1+z)$$

$$n\log(1+x) = \log(1+z)$$

Use these expressions in the left-hand side of the limit to obtain the right-hand side:

$$\frac{(1+x)^n - 1}{x} = \frac{z}{x}$$

$$= \frac{z}{x} \times \frac{\log(1+z)}{\log(1+z)}$$

$$= \frac{z}{\log(1+z)} \times \frac{n\log(1+x)}{x}$$

$$= \frac{1}{\log(1+z)^{\frac{1}{z}}} \times n\log(1+x)^{\frac{1}{x}}$$

Taking limit $z \to 0$, when $x \to 0$:

$$\lim_{x\to 0}\frac{(1+x)^n - 1}{x} = \lim_{z\to 0}\frac{1}{\log(1+z)^{\frac{1}{z}}} \times \lim_{x\to 0} n\log(1+x)^{\frac{1}{x}}$$

$$= \frac{1}{\log e} \times n \log e = n$$

7. $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

The given function $f(x) = \frac{\sin x}{x}$ is an even function, because

$$f(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x}$$

This means that we need only to find the right–hand limit, because the limiting behavior from the left will be the same as that of the right–hand limit. These values are shown in the following table:

x:	0.1	0.05	0.01	0.001	0
$f(x)$:	0.998334	0.999583	0.9999833	0.099999983	undefined

The table suggest that $\lim\limits_{x \to 0^+} \frac{\sin x}{x} = 1$. The limit is therefore $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

1.8 Continuous and Discontinuous Functions

i) *Recognition of left–hand and right-hand limits*

Definition 1.8.1: [Right–hand Limit of a Function]: A function $f(x)$ has a right–hand limit if $f(x)$ can be made as close to the number L as we please for all values of $x>c$:

$$\lim_{x \to c^+} f(x) = L \qquad (20)$$

Definition 1.8.2: [left–hand Limit of a function]: A function $f(x)$ has a left–hand limit if $f(x)$ can be made as close to the number L as we please for all values of $x<c$:

$$\lim_{x \to c^-} f(x) = L \qquad (21)$$

The relationship between one–sided and two–sided limits is summarized in the box:

The function has a limit as x approaches c if and only if, both the left–hand limit and the right–hand limit at c exist and are equal:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L \Rightarrow \lim_{x \to c} f(x) = L$$

Example 1.8.1: [Limit does not exist]: What is $\lim\limits_{x\to 0}\dfrac{|x|}{x}$?

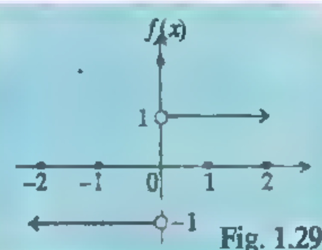


Fig. 1.29

The function $f(x)$ is not defined for $x=0$ (open hole). When $x>0$, the function is $f(x)=\dfrac{|x|}{x}=\dfrac{x}{x}=1$. When $x<0$, the function is $f(x)=\dfrac{|x|}{x}=-\dfrac{x}{x}=-1$. The graph of $f(x)$ is shown above. As x approaches 0 from the right, x is always positive and the corresponding value of $f(x)$ is 1. But as x approaches 0 from the left, x is always negative and the corresponding value of $f(x)$ is –1. Thus, as x approaches 0 from both sides, the corresponding values of $f(x)$ do not get closer and closer to a single real number. Therefore, the limit does not exist.

Example 1.8.2:[Limit does not exist]: How to determine the $\lim\limits_{x\to 1}\dfrac{1}{x-1}$?

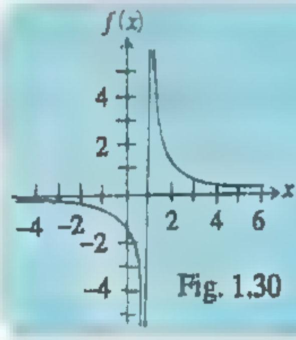


Fig. 1.30

Solution: The graph of a function $f(x)=\dfrac{1}{x-1}$ is shown in the Fig. 1.30.

The function $f(x)=1/(x-1)$ is not defined for $x=1$ (poles). However, the tabular form of the given function $f(x)=\dfrac{1}{x-1}$

x:	0	0.5	0.9	0.99	0.999	1	1.001	1.01	1.1	1.5	1
$f(x)$:	−1	−2	−10	−100	−1000	∞	1000	100	10	2	1

suggests that:

- as x approaches 1 from the left, the function $f(x)$ gets smaller and smaller. In fact $f(x)$ decreases without bound, and we write:

$$\lim_{x\to 1^-}\frac{1}{x-1}=-\infty$$

- as x approaches 1 from the right, the function $f(x)$ gets larger and larger. In fact $f(x)$ increases without bound, and we write:

$$\lim_{x\to 1^+}\frac{1}{x-1}=+\infty$$

The limit on the left is not the same as the limit on the right. Thus, the limit does not exist.

From the graphical view of a function, we can see that as the curve approaches 1, it does not approach the same functional value on both sides of $x=1$, the limit does not exist (since poles are developed). Also we note that the line $x=1$ is the vertical asymptote to the curve and $y=0$ is the horizontal asymptote to the curve.

The steps to define asymptotes, are the following:

1. If a number k makes the denominator of a rational function 0, but the numerator is nonzero, then the line $x=k$ is a vertical asymptote for the graph.
2. Whenever the values of y approach, but never equal, some number k as $|x|$ gets larger and larger, the line $y=k$ is a horizontal asymptote for the graph.
3. The horizontal asymptote of a rational function $y=\dfrac{ax+b}{cx+d}$ (with $(c\neq 0)$ is the

line $y=\dfrac{a}{c}$.

Example 1.8.3: [limit $x\to\infty$]: How to determine the $\lim\limits_{x\to\infty}\dfrac{3x^2+2}{x^2+1}$?

Solution: If the rational function is difficult to simplify, then we need to take out x^2 from numerator and denominator to obtain:

$$\lim_{x\to\infty}\frac{3x^2+2}{x^2+1}=\lim_{x\to\infty}\frac{x^2\left(3+\frac{2}{x^2}\right)}{x^2\left(1+\frac{1}{x^2}\right)}=\lim_{x\to\infty}\frac{3+\frac{2}{x^2}}{1+\frac{1}{x^2}}=\frac{3+\frac{2}{\infty}}{1+\frac{1}{\infty}}=\frac{3+0}{1+0}=3$$

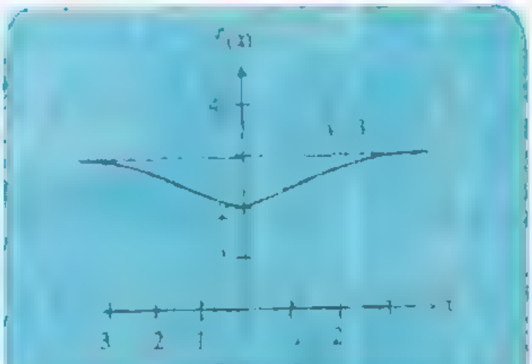


Fig 1.3.

From the graphical view of this function, we can see that the line $y=3$ is a horizental asymptote to the curve.

Example 1.8.4:[Estimating Limits by Graphing]: The graph of a function $f(x)$ is given below:

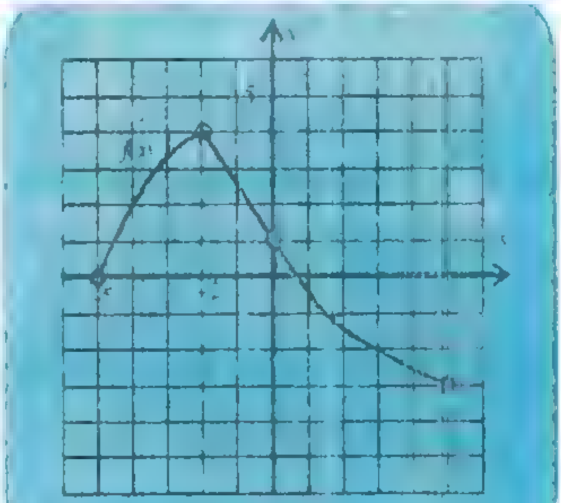


Fig. 1.32

Use this graph to indicate the following limits by inspection, if they exist:

a. $\lim\limits_{x\to 3^-} f(x)$ b. $\lim\limits_{x\to -2^+} f(x)$ c. $\lim\limits_{x\to 0} f(x)$ d. $\lim\limits_{x\to -4} f(x)$

Solution:

a. $\lim\limits_{x\to 3^-} f(x)$ is the value that $f(x)$ approaches as x tends towards 3 from the left. The figure indicates that this value is -2. Thus, we can write $\lim\limits_{x\to 3^-} f(x) = -2$.

b. $\lim\limits_{x\to -2^+} f(x)$ is the value that $f(x)$ approaches as x tends toward -2 from the right. The figure indicates that this value is 4 (which is related with a hole, this means that the point $(-2, 4)$ does not lie on this curve). So, the limit of a function at a hole exists and is therefore $\lim\limits_{x\to -2^+} f(x)=4$, but the function value at a hole is not equal to the limit of a function at $x=-2$. This is because of discontinuity of a function at $x=-2$. Thus, we can write $\lim\limits_{x\to -2^+} f(x)=4\neq f(-2)$.

c. $\lim\limits_{x\to 0} f(x)$ is the value that $f(x)$ approaches as x tends toward from both sides the left and the right. The figure indicates this value is 1 which is related with the hole. Thus, we can write $\lim\limits_{x\to 0^-} f(x)=1$ and $\lim\limits_{x\to 0^+} f(x)=1$, so $\lim\limits_{x\to 0} f(x)$ exists and

$\lim\limits_{x\to 0} f(x)=1$. But $\lim\limits_{x\to 0} f(x)\neq f(0)$..

d. $\lim\limits_{x\to -4} f(x)$ is the value that $f(x)$ approaches as x tends toward from both sides the left and right. The figure indicates this value is 2. Thus, we can write $\lim\limits_{x\to -4^-} f(x)=2$ and $\lim\limits_{x\to -4^+} f(x)=2$, so $\lim\limits_{x\to -4} f(x)$ exists and is therefore $\lim\limits_{x\to -4} f(x)=2$. The function value at $x=-4$ is also equal to the limit of a function at $x=-4$. This is because of continuity of the function at $x=-4$. Thus, we can write $\lim\limits_{x\to -4} f(x)=f(-4)=2$.

Remember, the function on which open hole occurs is valid to exist a limit. But the function with pole do not exist a limit. In each case, the function is always discontinuous.

ii) *Definition of continuity of a function at a point and in an interval*

A function is continuous at a point if you draw graph of the function near that point without lifting your pencil from the paper. Conversely, a function is discontinuous at a point if the pencil must be lifted from the paper in order to draw the graph on both sides of the point.

Definition 1.8.2:[Continuous Function]: A function $f(x)$ is said to be continuous at $x=c$, if all three of the following conditions are satisfied:

a. The function is defined at $x=c$; that is, $f(c)$ exists.

b. The function approaches a definite limit as x approaches c; that is $\lim\limits_{x\to c} f(x)$ exists.

c. The limit of a function is equal to the value of a function when x=c; that is,

$$\lim_{x \to c} f(x) = f(c)$$

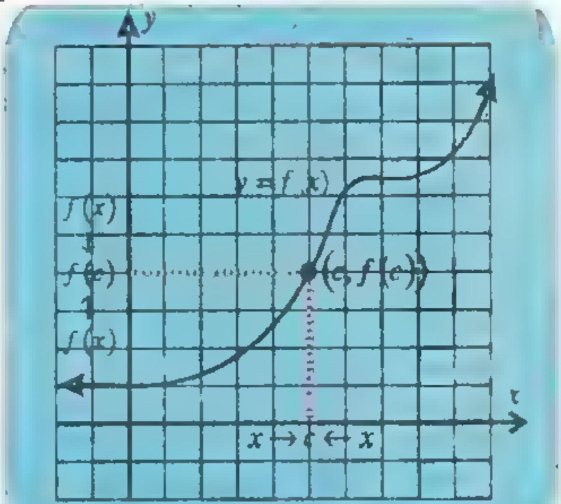


Fig. 1.33

Geometrically, this means that the point (x, f(x)) on the graph of $f(x)$ converge to the point (c, f(c)) as $x \to c$ and this is what guarantees that the graph is unbroken at (c, f(c)) with no "gap" or "open hole," as shown in the Fig. 1.33.

In other words, a function $f(x)$ is continuous on the open interval (a, b), if it is continuous at each point on the interval. If one or more of the three conditions in the definition fails, then the function is discontinuous at x=c. The three common ways for a function to be discontinuous at x=c, are the holes (open), poles, and jumps.

Example 1.8.5: [Continuous Function]: Use the definition of continuity to discuss the continuity of each of th following functions at the indicated points:

a. $f(x) = x+2$ at $x=2$

b. $g(x) = \dfrac{x^2-4}{x-2}$ at $x=2$

c. $h(x) = \dfrac{|x|}{x}$ at $x=0$ and at $x=1$

Solution:

a. The given function $f(x)$ is continuous at x=2, since

$$\lim_{x \to 2} f(x) = 2+2 = 4 = f(2)$$

This is shown in the figure (1.34(a)):

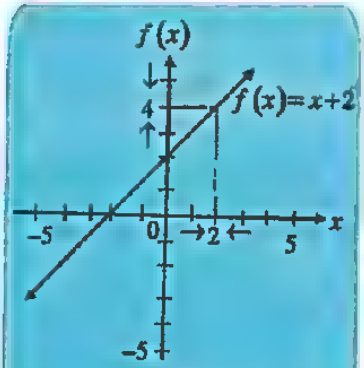


Fig. 1.34 (a)

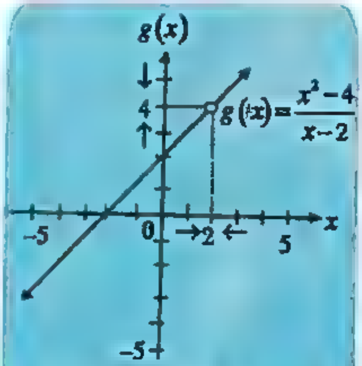


Fig. 1.34 (b)

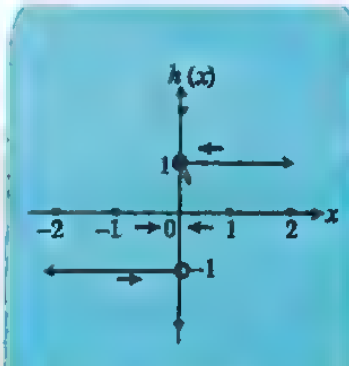


Fig. 1.34(c)

b. The given function g(x) is not continuous at x=2, since g(2)=0/0 is not defined. The discontinuity is shown by "open hole" on the graph of this function. This is shown in the figure (1.34(b)).

c. The function h(x) is not continuous at x=0, since h(0)=0/0 is not defined; also $\lim\limits_{x\to\infty} h(x)$ does not exist. But h(x) is continuous at x=1, since

$$\lim_{x\to 1}\frac{|x|}{x}=1=h(1)$$

This is shown in the Fig. 1.34(c). Functions have some useful general continuity properties. These properties enable us to determine intervals for some important classes of functions without having to look at their graphs or use the three conditions in the definition.

The properties of some specific functions are the following:

- A constant function $f(x)$=k, where k is a constant, is continuous for all x. For instance, the function $f(x)$=7 is continuous for all x.
- For n a positive integer, $f(x)=x^n$ is continuous for all x. For instance, the function $f(x)=x^5$ is continuous for all x.
- A polynomial function is continuous for all x. For instance, the function $2x^3-3x^2+x-5$ is continuous for all x.
- A rational function is continuous for all x except those values that make a denominator 0. For instance, the function $\frac{x^2+1}{x^2-1}$ is continuous for all x except x=1, a value that makes the denominator zero.

- For n an odd positive integer greater than 1, the function $\sqrt[n]{f(x)}$ is continuous whenever $f(x)$ is continuous. For instance, the function $\sqrt[3]{x^2}$ is continuous for all x.
- For n an even positive integer, the function $\sqrt[n]{f(x)}$ is continuous whenever $f(x)$ is continuous and non-negative. For instance, the function $\sqrt[4]{x}$ is continuous on the interval $[0,\infty)$.

Example 1.8.6:[Continuity Properties]: Use the general properties of continuity to discuss the continuity of the following functions:

a. $f(x)=x^2-2x+1$ b. $f(x)=\dfrac{x}{(x+2)(x-3)}$

c. $f(x)=\sqrt[3]{x^2-4}$ d. $f(x)=\sqrt{x-2}$

Solution:

a. $f(x)$ is continuous for all x, since $f(x)$ is a polynomial function.

b. $f(x)$ is continuous for all x except -2 and 3 (values that make the denominator 0), since $f(x)$ is a rational function,

c. The function $f(x)=\sqrt[3]{x^2-4}$ is continuous for all x. Since n=3 is odd, $f(x)$ is continuous for all x.

d. The function $f(x)=\sqrt{x-2}$ is continuous for all x and nonnegative for $x\geq 2$. Since n=2 is even, $f(x)$ is continuous for $x\geq 2$, or on the interval $[2,\infty)$.

iii) Determination of continuity and discontinuity of a function at a point and in an interval

Definition 1.8.7: [One–Sided Continuity]: If a and b are the endpoints of the interval, then the function $f(x)$ is said to be continuous from the right at a if and only if

$$\lim_{x\to a^+} f(x)=f(a) \qquad (22)$$

and is continuous from the left at b if and only if

$$\lim_{x\to b^-} f(x)=f(b) \qquad (23)$$

Example 1.8.7:[One–Sided Continuity]: First–class postage in 1995 was *\$0.32* for the first ounce (or any fraction thereof) and \$0.23 for each additional ounce (or

fraction thereof) up to 11 ounces. If p(x) is the amount of postage for a letter weighing in x ounces, then we write:

$$P(x)=\begin{cases} \$0.32, & \text{if } 0<x\le 1 \\ \$0.55, & \text{if } 1<x\le 2 \\ \$0.78, & \text{if } 2<x\le 3 \\ \text{and so on} \end{cases}$$

a. Graph p(x) for $0 < x \le 5$

b. Find $\lim\limits_{x\to 1^{-1}} p(x)$, $\lim\limits_{x\to 1^{+1}} p(x)$ and p(1).

c. Find $\lim\limits_{x\to 4.5} p(x)$ and p(4.5)

d. Find $\lim\limits_{x\to 4} p(x)$ and p(4).

e. Is p(x) continuous at x=1? x=4.5? x=4?

Solution

a. The graph of p(x) is shown in the figure (1.35).

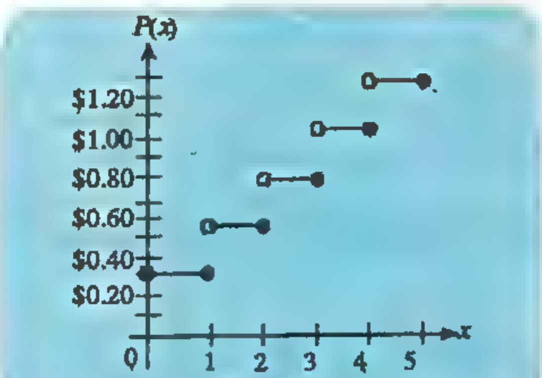


Fig. 1.35

b. From the graph of a function, the left, right limits and the value of a function at x=1 are:

$\lim\limits_{x\to 1^-} p(x)=0.32$, $\lim\limits_{x\to 1^+} p(x)=0.55$ and p(1)=0.32.

c. From the graph of the function, the limit and the value of function are equal:

$\lim\limits_{x\to 4.5} p(x)=1.24$, $p(4.5)=1.24$

Thus the function is continuous at x=4.5.

d. From the graph of a function, the limit and the value of a function are not equal:

$\lim_{x \to 4} p(x)$ does not exist and $p(4)=1.01$

Thus, the function is discontinuous at $x=4$

e. The function $p(x)$ is discontinuous at $x=1$ as well as at $x=4$, but continuous at $x=4.5$.

Sometimes functions need to be defined in pieces, because they have a split domain. These functions require more than one formula to define the function, and therefore are called **piecewise continuous functions.**

Exercise 1.4

1. Evaluate the following limits:

a. $\lim_{x \to -2}(x^2+3x-7)$ b. $\lim_{x \to 3}(x+5)(2x-7)$ c. $\lim_{z \to 1^-}\frac{z^2+z-3}{z+1}$

d. $\lim_{x \to 4}\left(\frac{1}{x}+\frac{3}{x-5}\right)$ e. $\lim_{x \to 1}\left(\frac{x^2+3x+2}{x^2+x+2}\right)^2$ f. $\lim_{x \to 1}\frac{\sqrt{x}-1}{x-1}$

g. $\lim_{x \to 0}\frac{\frac{1}{x+3}-\frac{1}{3}}{x}$ h. $\lim_{x \to 1}\frac{\frac{1}{x}-1}{x-1}$ i. $\lim_{x \to 0}\frac{1-\sin x}{\cos^2 x}$

j. $\lim_{x \to 0}\frac{\tan x}{x}$ k. $\lim_{x \to 0}\frac{\sec x-1}{x\sec x}$ l. $\lim_{x \to 0}\frac{\tan^2 x}{x}$

2. Use algebra and the rules of limits to evaluate the following limits :

a. $\lim_{x \to 4}\frac{-6}{(x-4)^2}$ b. $\lim_{x \to 0}\frac{\left(\frac{1}{(x+3)}\right)-\frac{1}{3}}{x}$

c. $\lim_{x \to 5}\frac{\sqrt{x}-\sqrt{5}}{x-5}$ d. $\lim_{x \to 25}\frac{\sqrt{x}-5}{x-25}$

3. Find the limit of the convergent of the following convergent sequences:

a. $\left\{\frac{5n}{n+7}\right\}$ b. $\left\{\frac{4-7n}{8+n}\right\}$ c. $\left\{\frac{8n-500\sqrt{n}}{2n+800\sqrt{n}}\right\}$

4. The weekly sales (in rupees) at a big store x weeks after the end of an advertising campaign are given by:

$$S(x)=5000+\frac{3600}{x+2}$$

Find the sale for the indicated weeks limits :

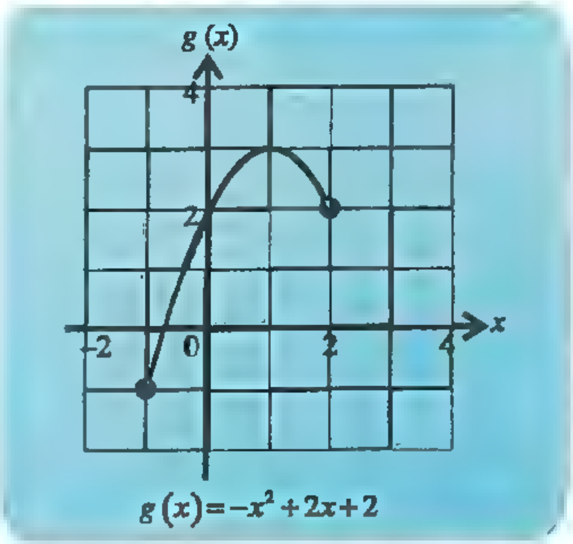
$a.\ S(2)$ $\quad b.\ \lim_{x\to 5} S(x)$ $\quad c.\ \lim_{x\to 16} S(x)$

5. Use properties of continuous function to test the continuity and discontinuity of the following functions:

a. $f(x) = 2x - 3$ b. $g(x) = 3 - 5x$ c. $h(x) = \dfrac{2}{x-5}$

d. $k(x) = \dfrac{x}{x+3}$ e. $g(x) = \dfrac{x-5}{(x-3)(x+2)}$ f. $F(x) = \dfrac{1}{x(x+7)}$

6. Use the graph of the function g(x) to answer the following questions:



a. Is g(x) continuous on the open interval (–1, 2)?

b. Is g(x) continuous from the right at x= –1? Is $\lim_{x\to -1^+} g(x) = g(-1)$?

c. Is g(x) continuous from the left at x=2? Is $\lim_{x\to 2^-} g(x) = g(2)$?

d. Is g(x) continuous on the closed interval [–1, 2]?

Glossary

- **Function:** A function $y=f(x)$ is a rule that assigns for each value of the independent variable x a unique value of the dependent variable y:

  $y = f(x)$

- **Compound Function:** A function defined by more than one equation is called a compound function.
- **Graph of a Function:** The graph of a function $f(x)$ consists of all points whose coordinates (x, y) satisfy a function $y=f(x)$, for all x in the domain of $f(x)$.
- **Inverse Functions:** Let $y=f(x)$ be a function of x. This function takes dependent variable y in response of independent variable x. The function that takes x as dependent variable in response of y as the independent is then called the inverse function of $f(x)$ and is denoted by:

  $x = f^{-1}(y)$

- **Algebraic Function:** A function $f(x)$ is called algebraic if it can be constructed using algebraic operations (such as adding, subtracting, multiplying, dividing, or taking roots) starting with polynomials. Any rational function is an algebraic function.
- **Transcendental Function:** Functions that are not algebraic are called transcendental
- **Explicit and Implicit Functions:** If a function is defined by an equation of the form $y=f(x)$, one says that the function is defined explicitly or is explicit. The terms "explicit function" and "implicit function" do not characterize the nature of the function but merely the way it is defined. Every explicit function $y=f(x)$ may also be represented as an implicit function $y-f(x)=0$
- **Parametric Equations]:** If $f(t)$ and $g(t)$ are continuous functions of parameter t on an interval D, then the equations

  $x=f(t)$ and $y=g(t)$

  are called the parametric equations for the plane curve C generated by the set of ordered pairs in plane:

  $(x\ )=(x(t), y(t))=(f(t), g(t))$

- **Limit of a function:** If $f(x)$ is a function of x, and c, L are the real numbers, then L is the limit of a function $f(x)$ as x approaches c :

  $\lim\limits_{x\to c} f(x) = L$

- **Continuous Function:** A function $f(x)$ is said to be a continuous at $x=c$, if all three of the following conditions are satisfied:
  - The function is defined at $x=c$; that is, $f(c)$ exists.
  - The function approaches a definite limit as x approaches c; that is $\lim\limits_{x\to c} f(x)$ exists.
  - The limit of a function is equal to the value of a function when $x=c$; that is,

    $\lim\limits_{x\to c} f(x)= f(c).$

UNIT 2

DIFFERENTIATION

This unit tells us how to:

- distinguish between independent and dependent variables.
- estimate the change in the dependent variable, when the independent variable is incremented or decremented.
- explain the concept of a rate of change.
- define the derivative of a function as an instantaneous rate of change of variable with respect to another variable.
- define derivative or differential coefficient of a function.
- differentiate $y=x^n$ and $y=(ax+b)^n$ by first principle rule.
- introduce the theorems of differentiation, such as the derivative of a constant function, the derivative of any constant multiple of a function, the derivative of a sum or difference of two functions, the derivative of the product of two functions and the derivative of a quotient of two functions.
- apply theorems of differentiation in solving problems.
- introduce chain rule of differentiation in different situations.
- introduce implicit differentiation of a function.
- introduce differentiation of trigonometric and inverse trigonometric functions.
- introduce differentiation of exponential and logarithmic functions.
- introduce the differentiation of hyperbolic and inverse hyperbolic functions.

2.1 Derivative of a Function

In this unit, the main ideas of differential calculus are under consideration. We begin by defining derivative, which is the central concept of differential calculus. Then we need to develop a list of rules and formulas for finding the derivative of a variety of expressions, including polynomial functions, rational functions, exponential functions, logarithmic functions, trigonometric functions and hyperbolic functions. The question is how the derivative can be interpreted as a rate of change.

i) Distinguish in between Independent and Dependent Variables

To understand the origin of the concept of variables, some real-life situations in which one numerical quantity depends on, corresponds to, or determines another are considered. For example,

1. The amount of income tax (output/dependent variable) you pay on the amount of your income (input/independent variable). The way in which the income determines the tax is given by the tax law (rule).
2. A person in business wants to know how profit (output/dependent variable) changes with respect to advertising (input/independent variable).
3. A person in medicine wants to know how a patient's reaction to a drug (output/dependent variable) changes with respect to dose (input/independent variable).

In each case, the change in dependent variable requires the definite change in independent variable through a definite rule which is called a **function.**

ii) Estimate the corresponding change in the dependent variable, when the independent variable is incremented or decremented

A familiar situation related to change in dependent with respect to change in independent is that a driver makes the run of 120, mile trip from Peshawar to Islamabad, in 2 hours. The table shows how far the driver has traveled from Peshawar at various times:

Time	0	0.5	1.0	1.5	2.00
Distance	0	24	54	88	120

If f is the function whose rule is **f(t)=distance from Peshawar** at time t, then, the table shows that $f(1.0)=54$, $f(1.5)=88$ and $f(2.0)=120$ miles. So the distance traveled from time $t=1.5$ to $t=2.0$ is $f(2)-f(1.5)=120-88=32$, the change in dependent variable (change in distance) in response of **incremented** independent variable t, while the distance traveled from time $t=1.5$ to $t=1.0$ is $f(1.5)-f(1.0)=88-54=35$, the change in dependent variable (change in distance) in response of **decremented** independent variable t.

iii) Concept of rate of change

The idea of average rate of change is something we encounter every day. For example, if a car accelerates from 0 to 96 km/h in 8.0 s, then we say that it accelerates at an average rate of 12 km/h. If a spaceship climbs from 0 to 10,000 m in 2.5 s,

then we say that the ship climbs at an average velocity of 4000 m/s. If corn grows a total of 28 inches in 2 weeks, then it grows an average of 2 inches per day.

In these examples, the indicated **average rate of change** is obtained by dividing the change in the **dependent variable** by the change in the **independent variable.**

Let us examine the process of finding the average rate of change of a function $y = f(x)$. If we select any value of x and increase it by an amount Δx, then a new value of the independent variable is $x+\Delta x$. As x changes from x to $x+\Delta x$, y will change to a corresponding amount of $y+\Delta y$. The ordered pairs $P(x, y)$ and $Q(x+\Delta x, y+\Delta y)$ developed must satisfy the function $y = f(x)$. This is shown in the figure (2.1):

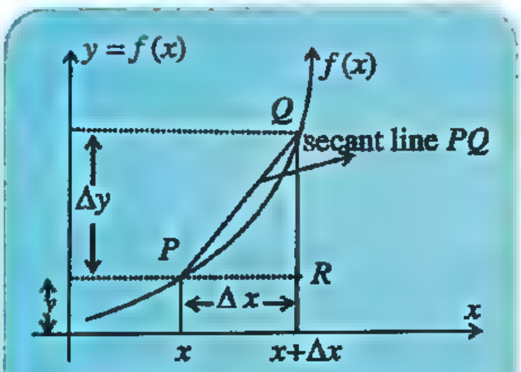


Fig. 2.1

If the function value at a point $P(x, y)$ is

$$y = f(x) \qquad (1)$$

then, the function value at a point Q is

$$y+\Delta y = f(x+\Delta x) \qquad (2)$$

The difference of equations (1) and (2) gives the change in y;

$$(y+\Delta y) - y = f(x+\Delta x) - f(x)$$
$$\Delta y = f(x+\Delta x) - f(x) \qquad (3)$$

The change in x is

$$\Delta x = x+\Delta x - x \qquad (4)$$

The average rate of change in y per unit change in x is the **slope of the secant line PQ**, obtained by taking the division of equation (3) by equation (4):

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{x+\Delta x - x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}, \quad y = f(x)$$

Definition 2.1.1: [Average Rate of Change]: The average rate of change y per unit change in x is given by:

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad (5)$$

The slope of the secant is the average rate of change which measures always "the approximate rate of change in phenomena."

Example 2.1.1: [Average Rate of Change]: Determine the average rate of change of y per unit change in x for $y=x^2-6x+5$ as x increases from $x=1$ to $x=3$.

Solution: The average rate of change through definition 3.1.1 is:

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}, \quad y=f(x)=x^2-6x+5$$

$$=\frac{[(x+\Delta x)^2-6(x+\Delta x)+5]-(x^2-6x+5)}{\Delta x}$$

$$=\frac{x^2+2x\Delta x+(\Delta x)^2-6x-6\Delta x+5-x^2+6x-5}{\Delta x} \qquad (6)$$

$$=\frac{2x\Delta x+(\Delta x)^2-6\Delta x}{\Delta x}=\frac{\Delta x(2x+\Delta x-6)}{\Delta x}=2x+\Delta x-6$$

The average rate of change (6) is used for $x=1$ and $\Delta x=2$ to obtain the average rate between 1 and 3:

$$\frac{\Delta y}{\Delta x}=2x+\Delta x-6$$

$$=2(1)+2-6=-2$$

Exercise 2.1

1. Find the average rate of change of the following functions over the indicated intervals:

a. $y=x^2+4$ from $x=2$ to $x=3$

b. $y=x^2+\frac{1}{3}x$ from x=-3 to x=3

c. $s=2t^3-5t+7$ from $t=1$ to t = 3

d. $h=\sqrt{2t}-7$ from $t=8$ to t = 8.5

2. Use definition 2.1.1 to find out the average rate of change over the specified interval for the following functions:

a. $s = 2t - 3$ from $t = 2$ to $t = 5$

b. $y = x^2 - 6x + 8$ from $x = 3$ to $x = 3.1$

c. $A = \pi r^2$ from $r = 2$ to $r = 2.1$

d. $h = \sqrt{t} - 9$ from $t = 9$ to $t = 16$

3. A ball is thrown straight up. Its height after t seconds is given by the formula $h = -16t^2 + 80t$. Find the average velocity $\frac{\Delta h}{\Delta t}$ for the specified intervals:

a. From t=2 to t=2.1. b. From t=2 to t=2.01.

4. The rate of change of price is called inflation. The price p in rupees after t years is $p(t) = 3t^2 + t + 1$. Find the average rate of change of inflation from $t = 3$ to $t = 5$ years. What the rate of change means? Explain.

iv) *Definition of derivative of a function as an instantaneous rate of change of a variable with respect to another variable*

In the previous sub-section, we discussed the average rate of change, and learned that the average rate of change is the slope of the secant line joining two points on the curve y=f(x). More commonly, we are asked to determine the exact or instantaneous rate of change at a particular time. For example, for an airplane, what is the instantaneous rate of change of the distance that occurs at a specific time? This can be dealt by the slope of a tangent line to a curve y=f(x) at a specific point?

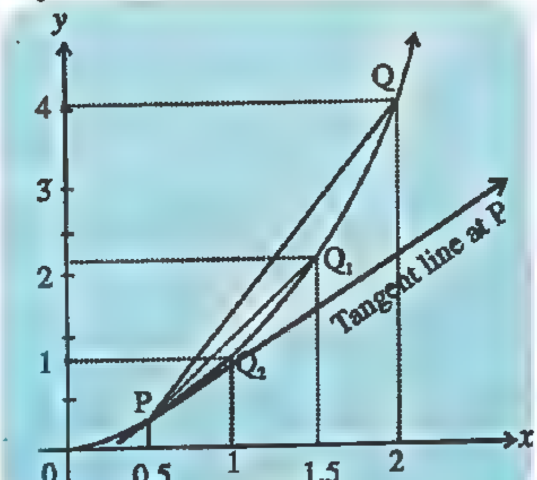


Fig. 2.2

To illustrate this idea, let us examine the graph of a function $y = x^2$ at a particular point P (0.5, 0.25) with different secant lines $PQ_1, PQ_2, \ldots$ that developed from the secant line PQ:

P	Q	Δx	Δy	$\frac{\Delta y}{\Delta x}$
P (0.5, 0.25)	$Q(2, 4)$	1.5	3.75	2.5
P (0.5, 0.25)	$Q_1(1.5, 2.25)$	1.0	2.00	2.0
P (0.5, 0.25)	$Q_2(1, 1)$	0.5	0.75	1.5
P (0.5, 0.25)	$Q_3(0.8, 0.64)$	0.3	0.39	1.3

The tabular form contains coordinates for the points P, Q, the change Δx in x, the change Δy in y and $\frac{\Delta y}{\Delta x}$, the slope of the secant lines $PQ, PQ_1, PQ_2, \ldots$ Notice that the slope of the secant line PQ is 2.5 ($\Delta y/\Delta x = 3.75/1.5 = 2.5$). If we take values of Q closer to P (i. e, to $Q_1, Q_2, Q_3, \ldots$), then, Δx gets smaller, and smaller, and tends to zero.

The tabular form clearly shows that, as Q approaches P, Δx approaches 0, and the slope of the secant line approaches the slope of the tangent line at a particular point P (0.5, 0.25) which is 1.

Geometrically, the **slope of the tangent line to a curve at a particular point P is the instantaneous (or exact) rate of change at that particular point.**

This terminology develops the idea that the slope of the secant line becomes a better approximation for the slope of the tangent line to the curve at a particular point P.

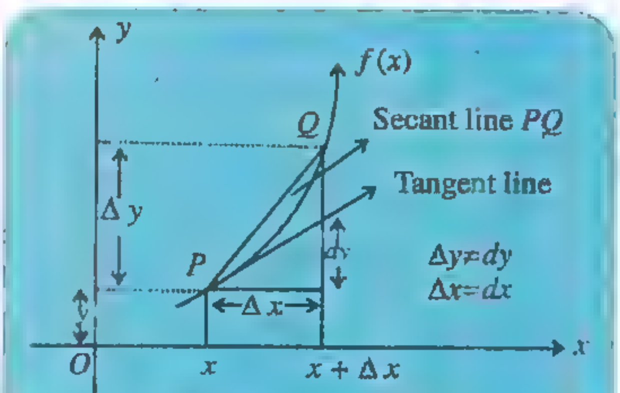


Fig. 2.3

From our discussion on limit, it follows that the exact/actual slope of the tangent line to a curve $y = f(x)$ at a particular point P corresponds to the instantaneous rate of change at that point. That is,

$$\frac{\Delta y}{\Delta x} = \text{slope of the secant line } PQ$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \text{slope of the tangent line at a particular point } P$$

The statement $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is read "the limit as delta x approaches zero of delta y divided by delta x." If the limit exists, then the result is the **slope of the tangent line or the instantaneous rate of change** of y with respect to x which we call the **derivative of the function.**

v) Definition of derivative or differential coefficient of a function

Definition 2.1.2: [Derivative of a Function]: The instantaneous rate of change of a function $f(x)$ at a point P is the derivative of a function $f(x)$ at that point P,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \text{ if this limit exists} \qquad (7)$$

This is called **first principle rule** of derivative of a function $f(x)$ with respect to x.

If y=f(x) is a function, then its derivative or differential coefficient is denoted by f' or y'. If x is a number in the domain of y=f(x) such that $y' = f'(x)$ is defined, then the function f is said to be differentiable at x. The process that produces the function f' from the function f is called differentiation.

Example 2.1.2: [First Principle Rule]: Determine the derivative of a function $f(x) = x^2 - 6x + 5$ by first principle rule at a point P (4, –3).

Solution: The derivative of a given function by first principle rule (7) is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x) = x^2 - 6x + 5$$

$$= \lim_{\Delta x \to 0} \frac{[(x+\Delta x)^2 - 6(x+\Delta x) + 5] - (x^2 - 6x + 5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 6x - 6\Delta x + 5 - x^2 + 6x - 5}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 - 6\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - 6)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 6) = 2x - 6$$

The result $f'(x) = 2x - 6$ represents the slope of the tangent line at any point $P\ (x,\ y)$ on the curve $f(x) = x^2 - 6x + 5$. Thus, the slope of the tangent line at a particular point, say $P\ (4, -3)$ on a given curve is:

$f'(x) = 2x - 6$

$f'(4) = 2(4) - 6 = 2,\ at\ P(4, -3)$

From this problem, we conclude that

- **the slope of the secant line (the average rate of change) is called the approximate rate of change.**
- **the slope of the tangent line (the instantaneous rate of change) is called the exact rate of change.**

First Principle Rule

If $f(x)$ is any function, then the derivative by first principle rule is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}, \quad y = f(x)$$

The process used for finding the derivative of a function in Example 2.1.2 is called the differentiation and its result,2, is the differential coefficient of a function $f(x) = x^2 - 6x + 5$ at a particular point P(4,-3).

Symbol $f'(x)$ is used to indicate the derivative of $f\ (x)$ with respect to x. Sometimes other symbols are used to indicate the derivative. Each of the symbols in the following box indicates the derivative of the dependent variable y with respect to the independent variable x:

1. $f'(x)$: read "f prime of x" (derivative of $f(x)$ with respect to x)
2. $\frac{dy}{dx}$: read " dee y, dee x" (the derivative of y with respect to x)
3. f' : read "f prime" (the derivative of the function $f(x)$ with respect to x)
4. $D_x\ y$: read " D sub x, y" (the derivative of y with respect to x)
5. y' : read "y prime" (the derivative of y with respect to x)

Definition 2.1.3: [Tangent line]: The tangent line to the graph of a function $y = f(x)$ at the point $(x, f(x))$ is the line through this point having slope

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad (8)$$

provided this limit exists. If this limit does not exist, then there is no tangent (no derivative) at the point.

The slope of the tangent line is the instantaneous rate of change, gives "the exact rate of change in the phenomena."

Example 2.1.3: The function is $f(x) = x^2$.

a. Find the derivative of a function at a point P(3,9)

b. Find the tangent line on a given curve $y = x^2$ at a point P(3,9).

c. View the slope of the tangent line on a curve $y = x^2$ at a point P (3,9) graphically.

Solution:

a. By first principle rule, the derivative of a given function is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x) = x^2$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - (x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x \quad (9)$$

b. Result (9) is used to obtain the slope of the tangent line at a P (3,9) on a curve $y = x^2$:

$$f'(3) = 2(3) = 6 \qquad (10)$$

The tangent line on a curve $y = x^2$ at a point $P(3,9)$ develops a nonhomogeneous line:

$y - y_1 = f'(x)(x - x_1)$, *Point form of the line*

$y - 9 = 6(x - 3)$, $P(3,9)$

$6x - y - 9 = 0$

c. The graphical view of the slope of the tangent line is as under:

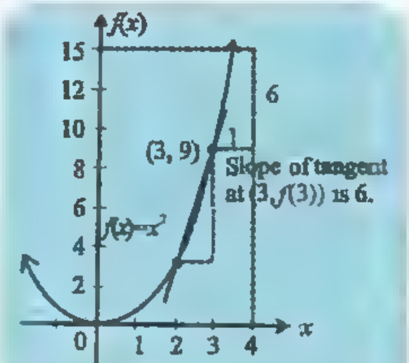


Fig. 2.4

vi) Differentiation of $y=x^n$ by first principle rule

If $f(x)=x^n, n$ is any rational number, then, by first principle rule, the derivative of $f(x)=x^n$ w.r.t x is:

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x},\quad y=f(x)=x^n$$

$$=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x},\ \textit{by binomial expansion}$$

$$=\lim_{\Delta x\to 0}\frac{x^n+nx^{n-1}\Delta x+\frac{n(n-1)}{2}x^{n-2}(\Delta x)^2+\ldots-x^n}{\Delta x},$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[nx^{n-1}\Delta x+\frac{n(n-1)}{2}x^{n-2}(\Delta x)^2+\ldots\right]$$

$$=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}\left[nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}\Delta x+\ldots\right]$$

$$=\lim_{\Delta x\to 0}[nx^{n-1}]+\lim_{\Delta x\to 0}\left[\frac{n(n-1)}{2}x^{n-2}\Delta x+\ldots\right]$$

$$=nx^{n-1}+0=nx^{n-1} \qquad (11)$$

vii) Differentiation of $y=(ax+b)^2$ by first principle rule

If $y=(ax+b)^2$, n is any rational number, then, the derivative of $y=(ax+b)^2$ is:

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}, \quad y=f(x)=(ax+b)^2$$

$$=\lim_{\Delta x\to 0}\frac{[a(x+\Delta x)+b]^2-(ax+b)^2}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{a^2(x+\Delta x)^2+2ab(x+\Delta x)+b^2-(a^2x^2+2abx+b^2)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{a^2(x^2+2x\Delta x+(\Delta x)^2)+2abx+2ab\Delta x+b^2-a^2x^2-2abx-b^2}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{(2a^2x+2ab)\Delta x+(\Delta x)^2}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\left[\frac{2(ax+b)\Delta x}{\Delta x}+\frac{(\Delta x)^2}{\Delta x}\right]$$

$$=2(ax+b)+0=2(ax+b)$$

This will help the students using power rule of differentiation.

Exercise 2.2

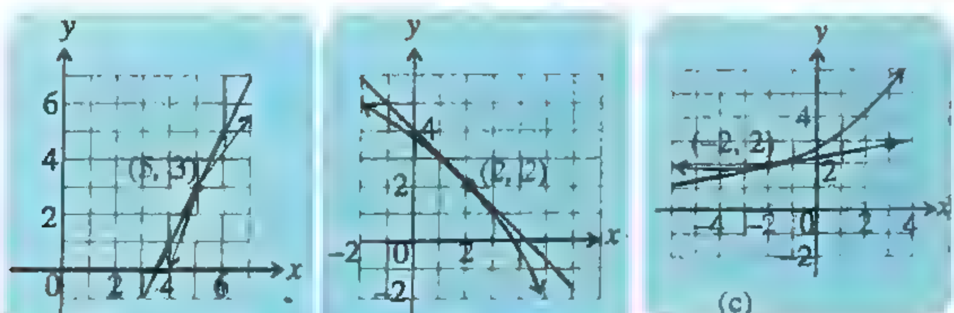
1. Use first principle rule to determine the derivative of the following functions:

a. $f(x)=3x$ b. $f(x)=5x+6$ c. $f(x)=x^2+1$

d. $f(x)=12-x^2$ e. $f(x)=16x^2-7x$ f. $f(x)=\frac{7}{x}$

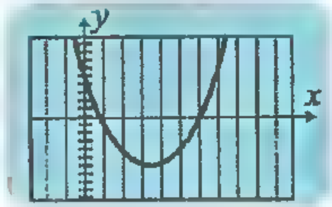
g. $f(x)=\frac{3}{x+3}$ h. $\frac{5}{2x-4}$ i. $f(x)=3x^2+4x-9$

2. Estimate the slope of the tangent line on a curve at a point $P(x, y)$ for each of the following graphs:



(a) (b) (c)

3. a. Draw a tangent line on the graph of the function $f(x)=x^2-7x+6$ at a point $(1, 0)$ of the function and estimate its slope at the same point graphically.



b. Draw a tangent line on the graph of a function $f(x)=x^2-7x+6$ at a point (5, -4) and estimate its slope at the same point graphically.

c. Is your estimation about the tangent lines at the points (1,0) and (5,-4) equal the actual derivatives of a function $f(x)=x^2-7x+6$ at the same points (1,0) and (5,-4)?

2.2 Theorem of Differentiation

In previous section, the derivative of a function f(x) was defined :

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

We learned that the derivative is found by applying the first principle rule. Now, after doing the exercise for the previous section, you may be wondering whether there is a shorter way of finding the derivative. In this and the next several sections, the discussion on the theorem that provides easier ways of finding derivatives.

Theorem 2.1:[Theorem of Differentiation]: If $f(x)$ and $g(x)$ are differentiable at all x, and a, b, and c are any real numbers, then the functions $cf(x)$, $f(x)\pm g(x)$, $f(x)g(x)$ and $f(x)/g(x)$ are also differentiable, and their derivatives satisfy the following rules:

Name of Rule	Leibnitz Notation
Constant rule	$\frac{d}{dx}(c)=0$
Constant multiple rule	$\frac{d}{dx}(cf)=c\frac{df}{dx}$
Sum rule	$\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx}$
Difference rule	$\frac{d}{dx}(f-g)=\frac{df}{dx}-\frac{dg}{dx}$

The constant multiple, sum and difference rules can be combined into a single rule, which is called the linearity rule.

Linearity rule $\qquad \frac{d}{dx}(af+bg)=a\frac{df}{dx}+b\frac{dg}{dx}$

Product rule $$\frac{d}{dx}(fg)=f\frac{dg}{dx}+g\frac{df}{dx}$$

Quotient rule $$\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{\frac{df}{dx}g-\frac{dg}{dx}f}{g^2}$$

The proofs of these rules in detail are as under:

- ***Proof Constant Rule***

If f(x)=c, where c is any constant, then, by first principle rule, the derivative of a constant function is:

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{c-c}{\Delta x}=0,\ f(x)=c$$

This calculation develops the rule that the derivative of a constant function is zero.

Constant Rule

If $f(x)=c$, where c is any constant, then:

$f'(x)=0$

Example 2.2.1: [Constant Rule]: Differentiate the following constant functions:

a. $f(x)=13$ b. $f(x)=3$ c. $f(x)=7\pi$ d. $f(x)=\sqrt{149}$

Solution: The graphs of the functions are horizontal lines parallel to x-axis, since all function are constant. The derivative in each case is therefore going to be zero.

- ***Proof Constant multiple rule***

If $f(x)=cx^n$, where c is any constant and n is a positive integer, then

by first principal rule, the derivative of a constant multiple function is:

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x},\quad y=f(x)=c\,x^n$$

$$=\lim_{\Delta x\to 0}\frac{c(x+\Delta x)^n-c\,x^n}{\Delta x}$$

$$=\lim_{\Delta x\to 0}c\times\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x}=c\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x}=cnx^{n-1}$$

This calculation develops the rule that the derivative of a constant multiple function is the product of the constant function and the derivative of a function f(x).

Constant Multiple Rule

If $g(x) = x^n$ and $f(x) = cg(x)$, c is any constant, then:

$f'(x) = cg'(x) = cnx^{n-1}$

Example 2.2.2: [Constant Multiple Rule]: Differentiate the following functions:

a. $f(x) = 4x^3$ b. $0.555x^6$

Solution:

a. If $f(x) = 4x^3$, then, the derivative of a given function is:

$f'(x) = 4(3)x^{3-1} = 12x^2$

b. If $f(x) = 0.555x^6$, then, the derivative of a given function is:

$f'(x) = 0.555(6)x^{6-1} = 3.33x^5$

- ***Proof Sum rule***

To determine the derivative of a polynomial, such as the derivative of the sum or difference of two or more functions, we need to develop a rule that could be used in the determination of a derivative like $f(x) = 3x^5 + 2x^2 + 3$. In this situation, if $h(x) = f(x) + g(x)$, then, our task is to determine $h'(x)$ by first principle rule of differentiation:

$$h'(x) = \lim_{\Delta x \to 0} \frac{[f(x+\Delta x) + g(x+\Delta x)] - [f(x) + g(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= f'(x) + g'(x)$$

This calculation develops the idea that the derivative of a sum is the sum of the derivatives of two functions. The difference of two functions f(x)-g(x) can be written as the sum of $f(x) - g(x) = f(x) + [-g(x)]$. Thus, the derivative of the difference of two functions is the difference of their derivatives.

Sum Rule

If $u = f(x)$ and $v = g(x)$, then, the sum rule can be restated using the notations:

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

This rule generalizes to the sum and difference of any given number of functions.

- ***Proof of Product rule***

If $h(x) = f(x)\,g(x)$, and f(x) and g(x) are differentiable functions of x, then by first principle rule, the derivative of h(x) is:

$$h'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \qquad (12)$$

The subtraction and addition of $f(x+\Delta x)g(x)$ to numerator of (12) is giving:

$$h'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)[g(x+\Delta x) - g(x)] + g(x)[f(x+\Delta x) - f(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} f(x+\Delta x)\left[\frac{g(x+\Delta x) - g(x)}{\Delta x}\right] + \lim_{\Delta x \to 0} g(x)\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} f(x+\Delta x) \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} g(x) \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= f(x)g'(x) + g(x)f'(x)$$

The $\lim_{\Delta x \to 0} f(x+\Delta x) = f(x)$, when Δx approaches 0. Also, $\lim_{\Delta x \to 0} g(x) = g(x)$, when Δx approaches 0.

This calculation develops the idea that the derivative of a product of two functions is the first function times the derivative of the second, plus the second function times the derivative of the first.

Product Rule

If $y = f(x)\,g(x) = u\,v$ with $u = f(x)$ and $v = g(x)$, then the product rule can be restated using the notations:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

- ***Proof Quotient rule***

If $h(x)=\frac{f(x)}{g(x)}$, $g(x)\neq 0$ and $f(x)$ and $g(x)$ are differentiable functions of x, then, the derivative of h(x) can be found by first principle rule:

$$h'(x)=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left\{\frac{f(x+\Delta x)}{g(x+\Delta x)}-\frac{f(x)}{g(x)}\right\}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left\{\frac{f(x+\Delta x)g(x)-f(x)g(x+\Delta x)}{g(x+\Delta x)g(x)}\right\} \qquad (13)$$

$$h'(x)=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left\{\frac{f(x+\Delta x)g(x)-f(x)g(x+\Delta x)}{g(x+\Delta x)g(x)}\right\}$$

The subtraction and addition of $f(x)\,g(x)$ to numerator of (13) is giving:

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left\{\frac{f(x+\Delta x)g(x)-f(x)g(x)-f(x)g(x+\Delta x)+f(x)g(x)}{g(x+\Delta x)g(x)}\right\}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left\{\frac{[f(x+\Delta x)-f(x)]g(x)-[g(x+\Delta x)-g(x)]f(x)}{g(x+\Delta x)g(x)}\right\}$$

$$=\lim_{\Delta x\to 0}\left\{\frac{\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]g(x)-\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]f(x)}{g(x+\Delta x)g(x)}\right\}$$

$$=\frac{\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]g(x)-\lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]f(x)}{\lim_{\Delta x\to 0}[g(x+\Delta x)g(x)]}$$

$$=\frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2}$$

Quotient Rule

If $y=f(x)/g(x)=u/v$, $v\neq 0$, with $u=f(x)$ and $v=g(x)$, then the quotient rule can be restated using the notations:

$$\frac{dy}{dx}=\frac{\frac{du}{dx}v-\frac{dv}{dx}u}{v^2}$$

Example 2.2.3: [Sum/Product/Quotient Rules]: Differentiate the following functions:

a. $y=4x^3-2x^2+5x$ b. $y=(x^2-2x)(x^3-3)$ c. $y=\dfrac{x^2+13x+9}{x^2+11x+3}$

Solution:

a. If $y=u+v+w=(4x^3)-(2x^2)+(5x)$, then the derivative of a given function is.

$$\frac{dy}{dx}=\frac{d}{dx}(u+v+w)$$

$$=\frac{d}{dx}[(4x^3)-(2x^2)+(5x)]$$

$$=\frac{d}{dx}(4x^3)-\frac{d}{dx}(2x^2)+\frac{d}{dx}(5x)=12x^2-4x+5$$

b. If $y=uv=(x^2-2x)(x^3-3)$, then, the derivative of a given function is:

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$=(x^2-2x)(3x^2)+(x^3-3)(2x-2)$$

$$=3x^4-6x^3+2x^4-2x^3-6x+6=5x^4-8x^3-6x+6$$

c. If $y=\dfrac{u}{v}=\dfrac{x^2+13x+9}{x^2+11x+3}$ then, the derivative of a given function is:

$$\frac{dy}{dx}=\frac{\frac{du}{dx}v-\frac{dv}{dx}u}{v^2}$$

$$=\frac{(2x+13)(x^2+11x+3)-(2x+11)(x^2+13x+9)}{(x^2+11x+3)^2}=\frac{-2(x^2+6x+30)}{(x^2+11x+3)^2}$$

2.3 Applications of Differentiation Theorem

Example 2.3.1: The cost in (million) dollars to produce x units of wheat is given by $C(x)=5000+20x+10\sqrt{x}$. Find the marginal cost, when

a. $x=9$ units b. x = 16 units

c. As more wheat is produced, what happens to the marginal cost?

Marginal Analysis: In business and economics the rates of change of such variables as cost, revenue and profit are most important. Economists use the word **marginal** to refer to rates of change. For example, marginal cost refers to the rate of change of

cost. Since the derivative of a function gives the rate of change of the function, a marginal cost (or revenue of profit) function is found by taking the derivative of the cost ((or revenue of profit) function. The marginal cost at some level of production x is the cost to produce the (x+1)st item (i.e., one more item).

If $C(x)$ is the cost of producing x units of some item, then the cost of producing x+1 units is $C(x+1)$. The actual cost of the x+1 unit is therefore $\frac{C(x+1)-C(x)}{x+1-x}=C(x+1)-C(x)$.

Solution:

a. To find the marginal cost, find the derivative of $C(x)=5000+20x+10\sqrt{x}$ with respect to x:

$$C'(x)=20+10(\frac{1}{2})(x^{-\frac{1}{2}})=20+\frac{5}{\sqrt{x}}$$

The approximate cost of x=9 units are:

$$C'(9)=20+\frac{5}{\sqrt{9}},\ x=9$$

$$=20+\frac{5}{3}=\frac{65}{3}=21.67\ dollars$$

The actual cost to produce 1 more unit is:

$C(x+1)-C(x)=C(10)-C(9)=5231.62-5210=21.62.$

The actual cost is 21.62 dollars.

b. The approximate cost x=16 units are:

$$C'(16)=20+\frac{5}{\sqrt{16}}$$

$$=20+\frac{5}{4}=\frac{85}{4}=21.25\ dollars$$

The actual cost to produce 1 more unit is:

$C(x+1)-C(x)=C(17)-C(16)=5381.23-5360=21.23$

The actual cost is 21.23 dollars.

c. It decreases and approaches 20$.

In business and economics terminologies,

- the instantaneous rate of change of cost (or revenue or profit) is the marginal cost which treats "as the approximate rate of change in business/ economics phenomena ".
- the average rate of change of cost is the "exact/actual rate of change in business/ economics phenomena."

Exercise 2.3

1. Use the product rule to find out the derivative of the following functions:

a. $y=(x^2-2)(3x+1)$ b. $y=(6x^3+2)(5x-3)$

c. $y=(7x^4+2x)(x^2-4)$ d. $y=(2x^2+4x-3)(5x^3+x+2)$

e. $y=(2x-3)(\sqrt{x}-1)$ f. $y=(-3\sqrt{x}+6)(4\sqrt{x}-2)$

2. Use the quotient rule to find out the derivative of the following functions:

a. $y=\dfrac{3x-5}{x-4}$ b. $y=\dfrac{2}{3x-5}$ c. $f(t)=\dfrac{t^2+t}{t-1}$

d. $y=\dfrac{-x^2+6x}{4x^3+1}$ e. $y=\dfrac{5x+6}{\sqrt{x}}$ f. $y=\dfrac{x^2+7x-2}{x-2}$

g. $f(p)=\dfrac{(2p+3)(4p-1)}{(3p+2)}$ h. $g(x)=\dfrac{x^3+1}{(2x+1)(5x+2)}$

3. Find an equation of a tangent line to the graph of the function at the particular point in the following problems:

a. $f(x)=3x-7$ at (3, 2) b. $f(x)=x^3$ at $x=-1/2$

c. $f(x)=\dfrac{1}{x+3}$ at $x=2$ d. $f(x)=\dfrac{x}{x-2}$ at (3,3)

4. A company that manufactures bicycles has determined that a new employee can assemble

$$M(d)=\frac{200d}{3d+10}$$

bicycles per day after d days of on-the-job training.

a. Find $M'(d)$. b. Find and interpret $M'(2)$ and $M'(5)$.

5. Suppose that the temperature T of food placed in a freezer drops according to the equation

$$T=\frac{700}{t^2+4t+10},$$ where t is the time in hours.

Determine the rate of change of T with respect to t for the following

a. $t=1$hr b. 2hr

6. For a thin lens of constant focal length P, the object distance x and the image distance y are related by the formula

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{P}$$

a. Solve the above equation for y in terms of x and P.

b. Determine the rate of change of y with respect to x.

7. Suppose you are the manager of a trucking firm, and one of your drivers reports that, according to her calculations, her truck burns fuel at the rate of

$$G(x)=\frac{1}{200}\left(\frac{800}{x}+x\right),\quad G'(x)=\frac{1}{200}\left(-\frac{800}{x^2}+1\right)$$

gallons per mile when traveling at x miles per hour on a smooth dry road.

a. If the driver tells you that she wants to travel 20 miles per hour, what should you tell her?

b. If the driver wants to go 40 miles per hour, what should you say?

2.4 Chain Rule

Suppose you are asked to find the derivative of

$h(x)=\ln(2x+1)$ or $h=\sqrt{x^3}$

We have formulas for computing derivatives of $\ln x$ and x^3, but not in the indicated combinations. The chain rule is used to compute derivatives of functions that are compositions of more elementary functions whose derivatives are known. The chain rule is therefore considered with a brief review of above composite functions.

Consider the function h (x) whose rule is $h(x)=\sqrt{x^3}$. To compute $h(4)$, you first find $x^3=4^3=64$ and then take the square root $\sqrt{x^3}=\sqrt{64}=8$ The rule of h (x) may be rephrased as:

$$h(x)=f(g(x))$$

where $g(x)=x^3$ and $f(x)=\sqrt{x^3}$. We may think of the functions f (x) and g (x) as being "composed" to create the function h.

To see this more clearly, If

$y=f(u)=\sqrt{u}$ and $u=g(x)=x^3$

then we can express y as a function of x as follows:

$$y=f(u)=f(g(x))=\sqrt{x}=h(x)$$

The function $h(x)$ is said to be the **composite** of the two simpler functions $f(x)$ and $g(x)$. (Roughly speaking, we can think of $h(x)$ as a function of a function.) Here is a formal definition of this idea.

Definition 2.4.1: [Composite Function]: A function h(x) is a composite of functions $f(x)$ and $g(x)$ if

$$h(x) = f(g(x))$$

The domain of $h(x)$ is the set of all numbers x such that x is in the domain of $g(x)$ and $g(x)$ is in the domain of $f(x)$.

Example 2.4.1: [Composite Functions]: Let $f(u) = e^u, g(x) = 3x^2 + 1$, and $m(v) = v^{3/2}$. Find the following composite functions:

a. $f(g(x))$ b. $g(f(u))$ c. $m(g(x))$

Solution:

a. $f(g(x)) = e^{g(x)} = e^{3x^2+1}$

b. $g(f(u)) = 3(f(u))^2 + 1 = 3(e^u)^2 + 1 = 3e^{2u} + 1$

c. $m(g(x)) = (g(x))^{3/2} = (3x^2+1)^{3/2}$

i) *Differentiation of composite functions:* $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

The word chain in the name chain rule comes from the fact that a function formed by composition (such as those in Example 3.4.1) involves a chain of functions (that is, a function of a function). The chain rule will enable us to compute the **derivative of a composite function in terms of the derivatives of the functions making up the composition.**

Suppose

$$y = h(x) = f[g(x)] \quad (14)$$

is a composite of f and g, where

$$y = f(u) \text{ and } u = g(x)$$

The derivative dy/dx of (14) in terms of the derivatives of $f(x)$ and $g(x)$ by first principle rule is:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x}$$

Substitute $h(x+\Delta x) = f[g(x+\Delta x)]$, $h(x) = f[g(x)]$ to obtain:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f[g(x+\Delta x)] - f[g(x)]}{\Delta x} \qquad (15)$$

Multiply and divide out (15) by $g(x+\Delta x) - g(x)$ to obtain :

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[\frac{[f[g(x+\Delta x)] - f[g(x)]]}{\Delta x} \left(\frac{g(x+\Delta x) - g(x)}{g(x+\Delta x) - g(x)} \right) \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{[f[g(x+\Delta x)] - f[g(x)]]}{g(x+\Delta x) - g(x)} \left(\frac{g(x+\Delta x) - g(x)}{\Delta x} \right) \right] \qquad (16)$$

The second factor in (16) is the difference quotient of $g(x)$. To interpret the first factor in (16) as the difference quotient of $f(u)$, if we put $k = g(x+\Delta x) - g(x)$, since u=g(x), then, we can write

$$u + k = g(x) + g(x+\Delta x) - g(x) = g(x+\Delta x)$$

Substitute this in (16) to obtain:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(u+k) - f(u)}{k} \left(\frac{g(x+\Delta x) - g(x)}{\Delta x} \right) \qquad (17)$$

If $k = [g(x+\Delta x) - g(x)] \to 0$ as $\Delta x \to 0$, then we can find the limit of each difference quotient in (17):

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[\left(\frac{f(u+k) - f(u)}{k} \right) \left(\frac{g(x+\Delta x) - g(x)}{\Delta x} \right) \right]$$

$$= \left[\lim_{k \to 0} \frac{f(u+k) - f(u)}{k} \right] \left[\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \qquad (18)$$

$$= f'(u)\, g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}$$

This result is correct under rather general conditions, and is called the **chain rule**, but our "derivation" is superficial, because it ignores a number of hidden problems.

ii) *Differentiation of a function of a function:* $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Theorem: 2.2: [Chain Rule]: If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then a composite function $y = f[g(x)]$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)]\frac{d}{dx}[g(x)] = \frac{dy}{du}\frac{du}{dx}$$

Proof:

Suppose x is changed by a small amount Δx. This will cause u to change by an amount Δu, which in turn, will cause y to change by an amount Δy. If Δu is not zero, then we can write

$$\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} \qquad (19)$$

By letting $\Delta x \to 0$, we force Δu to approach zero as well, because

$$\Delta u=\left(\frac{\Delta u}{\Delta x}\right)\Delta x \quad so \quad \lim_{\Delta x\to 0}\Delta u=\left(\frac{du}{dx}\right)(0)=0$$

With this assumption, the equation (19) takes the form:

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\left(\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\right)$$

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx} \qquad (20)$$

This can easily be extended to compositions of three or more functions. For example, if $y=f(u)$, $u=g(v)$ and $v=h(x)$, then:

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dv}\frac{dv}{dx} \qquad (21)$$

Example 2.4.2: [Chain Rule]: Differentiate the following functions w.r.t. x:

a. $f(x)=(4x-3)^3$ b. $f(x)=\sqrt{15x^2+1}$

Solution:

a. If $y=f(x)=(4x-3)^3=f(u)=u^3$ with $u=4x-3$, then, the first derivative w.r.t x by chain rule is:

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$

$$=\frac{d}{du}(u^3)\frac{d}{dx}(4x-3)=(3u^2)(4)=12u^2=12(4x-3)^2, u=(4x-3)$$

b. If $y=f(x)=\sqrt{15x^2+1}=f(u)=\sqrt{u}=u^{\frac{1}{2}}$ with $u=15^2+1$, then, the first derivative by chain rule is:

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}, \quad u=(15x^2+1)$$

$$=\frac{d}{du}(u^{1/2})\frac{d}{dx}(15x^2+1)=\frac{1}{2}(u^{\frac{1}{2}-1})(30x)=\frac{15x}{\sqrt{u}}=\frac{15x}{\sqrt{15x^2+1}}$$

Example 2.4.3: The revenue realized by a small city from the collection of fines from parking tickets is given by

$$R(x)=\frac{8000x}{x+2}$$

where x is the number of work hours each day that can be devoted to parking patrol. At the outbreak of a flu epidemic, 30 work hours are used daily in parking patrol, but during the epidemic that number is decreasing at the rate of 6 work hours per day How fast is revenue from parking fines decreasing during the epidemic?

Solution: We need to find dR/dt, the change in revenue with respect to time t. The chain rule is used to obtain dR/dt :

$$\frac{dR}{dt}=\frac{dR}{dx}\frac{dx}{dt} \tag{22}$$

First find dR/dx as follows.

$$R'(x)=\frac{dR}{dx}=\frac{(x+2)((8000)-8000x(1)}{(x+2)^2}=\frac{16000}{(x+2)^2}$$

$$R'(30)=\frac{16000}{(30+2)^2}=15.625, \quad at\ x=30$$

dR/dx=15.625 and $dx/dt=-6$ are used in equation (22) to obtain:

$$\frac{dR}{dt}=\frac{dR}{dx}\frac{dx}{dt}=(15.625)(-6)=-93.75$$

This tells us that the revenue is being lost at the rate of approximately \$94 per day.

iii) *Differentiation of inverse functions:* $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$

Let $x=f(y)$ be a one-to-one differentiable function. The inverse function of a function $x=f(y)$ is $y=f^{-1}(x)$. If $x=f(y)$ is differentiable at $y=f^{-1}(x)$ and $f'(y)=f'\left(f^{-1}(x)\right)$ is not equal zero, then $y=f^{-1}(x)$ is differentiable at x and leads

the differentiation formula:

$$\frac{d}{dx}f^{-1}(x)=\frac{1}{f'\left(f^{-1}(x)\right)},\ x=f(y)$$

$$\frac{dy}{dx}=\frac{1}{f'(y)} \qquad (23a)$$

$$=\frac{1}{\frac{dx}{dy}},\ \textit{Leibnitz's Notation}$$

In case of a function $y=f(x)$, the differentiation formula (23a) becomes:

$$\frac{d}{dy}f^{-1}(y)=\frac{1}{f'\left(f^{-1}(y)\right)},\ y=f(x)$$

$$\frac{dx}{dy}=\frac{1}{f'(x)} \qquad (23b)$$

$$=\frac{1}{\frac{dy}{dx}},\ \textit{Leibnitz's Notation}$$

Example 2.4.4: [$\frac{dy}{dx}=1/(dx/dy)$]: Verify result $\frac{dy}{dx}=1/\frac{dx}{dy}$ for the following problems:

a. $f(x)=(4x-3)^3$ b. $f(x)=\sqrt{15x^2+1}$

Solution:

a. The derivative of $y=(4x-3)^3$ is $\frac{dy}{dx}=12(4x-3)^2$. This result agrees to result (23a):

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\frac{1}{12(4x-3)^2}}=(1)\left(\frac{12(4x-3)^2}{1}\right)=12(4x-3)^2$$

b. $y=\sqrt{15x^2+1}$ is $\frac{dy}{dx}=\frac{15x}{\sqrt{15x^2+1}}$. This result agrees to result (23a):

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\frac{1}{\frac{\sqrt{15x^2+1}}{15x}}}=(1)\left(\frac{15x}{\sqrt{15x^2+1}}\right)=\frac{15x}{\sqrt{15x^2+1}}$$

(iv) Generalized power rule: $\frac{dy}{dx}=n[f(x)]^{n-1}\frac{d}{dx}f(x)$

Can we find the derivative of $f(x)=(4x-3)^2$? The answer is yes, if we expand the given function by binomial to obtain

$$f(x)=(4x-3)^2=16x^2-24x+9,$$

then the derivative of f(x) is

$$f'(x)=32x-24 \qquad (24)$$

The derivative of $f(x)=(4x-3)^3$ is obtained by expanding the given function again by binomial

$$f(x)=(4x-3)^3=64x^3-144x^2+108x-27,$$

then the derivative of f(x) is

$$f'(x)=192x^2-288x+108 \qquad (25)$$

However, when we try to find the derivative of $f(x)=(4x-3)^6$, the process of binomial expansion is time consuming task. The easier way other than the expansion of binomial is the generalized power rule.

For generalized power rule, we need to re-examine the result (24)

$$\begin{aligned} f(x)&=(4x-3)^2, && \textit{Original function}\\ f'(x)&=32x-24, && \textit{Derivative of a function}\\ &=2(16x-12), && \textit{Cmmon factor 2}\\ &=2(4x-3)4, && \textit{Common factor of 4}\end{aligned}$$

that leads the special notation

$$f(x)=[g(x)]^2, \qquad g(x)=4x-3$$

$$f'(x)=2[g(x)]4 \qquad (26)$$

$$=2[g(x)]g'(x), \quad g'(x)=4$$

Similarly, the reexamination of result (25)

$$f(x)=(4x-3)^3, \quad \textit{Original function}$$
$$f'(x)=192x^2-288x+108, \quad \textit{Derivative of the function}$$
$$=3(64x^2-96x+36), \quad \textit{Common factor 2}$$
$$=3(4x-3)^2 4, \quad \textit{Common factor 4}$$

- that leads the special notation

$$f(x)=[g(x)]^3, \qquad g(x)=4x-3 \qquad (27)$$
$$f'(x)=3[g(x)]^2 4=3[g(x)]^2 g'(x), \quad g'(x)=4$$

These results develop the general term $f'(x)=n[g(x)]^{n-1}g'(x)$, when $f(x)=[g(x)]^n$. In words, if f(x) is equal to an expression in x raised to a power of n, then $f'(x)$ is equal to the product of n times the expression to the n-1 power times the derivative of the expression with respect to the variable. The statement is known as **the general power rule.**

Definition 2.4.2:[Generalized Power Rule]: If $f(x)=[g(x)]^n$ and $g(x)$ is a differentiable function of x, then

$$f'(x)=n[g(x)]^{n-1}.g'(x), \qquad \text{where n is a real number.}$$

General Power Rule

If $y=[f(x)]^n$, and $f(x)$ **is a differentiable function of x,** then:

$$\frac{dy}{dx}=n[f(x)]^{n-1}\frac{d}{dx}f(x), \text{ n is a real number.}$$

Example 2.4.5: [General Power Rule]: Differentiate the following functions:

a. $f(x)=(11x^2-7)^8$ b. $f(x)=\sqrt{2x^3+11}$

Solution:

a. If $y=f(x)=(11x^2-7)^8$ then, the first derivative of a given function is:

$$\frac{dy}{dx}=n[f(x)]^{n-1}\frac{d}{dx}[f(x)]$$
$$=8(11x^2-7)^{8-1}\frac{d}{dx}(11x^2-7), \quad n=8, f(x)=(11x^2-7)$$
$$=8(11x^2-7)^7(22x)=176x(11x^2-7)^7$$

b. If $y = f(x) = \sqrt{2x^3 + 11}$, then, the first derivative of a given function is:

$$\frac{dy}{dx} = n[f(x)]^{n-1}\frac{d}{dx}[f(x)]$$

$$= \frac{1}{2}(2x^3 + 11)^{\frac{1}{2}-1}\frac{d}{dx}(2x^3 + 11), \quad n = 1/2,\ f(x) = 2x^3 + 11$$

$$= \frac{1}{2}(2x^3 + 11)^{-\frac{1}{2}}(6x^2) = \frac{3x^2}{\sqrt{2x^3 + 11}}$$

v) *Derivative of a parametric function*

Consider two differentiable functions $x = f(t)$ and $y = g(t)$ of parameter t. If $t = h(x)$ is an inverse function of $x = f(t)$, then

$$y = g[h(x)]$$

is a function of x.

By chain rule, the differentiation of $y = g[h(x)]$ w.r.t x is

$$\frac{dy}{dx} = \frac{dy}{dx}\frac{dt}{dt}, \text{ multiply and divide out by } dt$$

$$= \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{\frac{dx}{dt}} = \frac{dy}{dt}\frac{dt}{dx} = \frac{g'(t)}{f'(t)} \qquad (28)$$

Example 2.4.6:[Parametric Function]: Find dy/dx, when $(x, y) = (at^2, 2at)$.

Solution: The chain rule for dy/dx is:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{\frac{dx}{dt}} = (2a)\left(\frac{1}{2at}\right) = \frac{1}{t}, \quad \frac{dy}{dt} = 2a,\ \frac{dx}{dt} = 2at$$

1 Determine the indicated derivative in each case:

a. $s = 5(7-t)^4, \frac{ds}{dt} = ?$

b. $w = 4(x^3 - 4x + 2)^5, \frac{dw}{dx} = ?$

c. $x = -3(4 - 11s^2)^5, \frac{dx}{ds} = ?$

d. $y = \frac{(4 - x^3)^{11}}{5}, \frac{dy}{dx} = ?$

e. $u = \sqrt[3]{1 - 3t^2}, \frac{du}{dt} = ?$

f. $s = \frac{1}{(3t+1)^7}, \frac{ds}{dt} = ?$

g. $R=\dfrac{1}{(2x-1)^{8}},\ \dfrac{dR}{dx}=?$ h. $R=\dfrac{1}{5(4x^{2}-7)^{7}},\ \dfrac{dR}{dx}=?$

2. Determine the derivative $f'(x)$ in each case:

a. $f(x)=(2x-5)^{3}(5x-7)$ b. $f(x)=\dfrac{(x+2)^{2}}{x-1}$

c. $f(x)=\left(\dfrac{2x-5}{x-4}\right)^{4}$ d. $f(x)=x\sqrt{2x^{2}+11}$

e. $f(x)=3x\sqrt[3]{3x+7}$ f. $f(x)=\dfrac{\sqrt{2x+11}}{(3x-8)^{2}}$

g. $f(x)=\left(\dfrac{3x-8}{x+9}\right)^{7}$ h. $f(x)=(2x-9)^{2}\sqrt{3x+7}$

3. Find dy/dx of the following function in terms of parameter t:

a. $x=1+t^{2},\ y=t^{3}+2t^{2}+1$ b. $x=3at^{2}+2,\ y=6t^{4}+9$

c. $x=\dfrac{a(1-t^{2})}{1+t^{2}},\ y=\dfrac{2bt}{1+t^{2}}$ d. $x=\dfrac{3at}{1+t^{3}},\ y=\dfrac{3at^{2}}{1+t^{3}}$

4. Studies show that after t hours on the job, the number of items a supermarket cashier can ring up per minute is given by

$$F(t)=60-\frac{150}{\sqrt{8+t^{2}}}$$

a. Find $F'(t)$, the rate at which the cashier's speed is increasing.

b. At what rate is the cashier's speed increasing after 5 hours? After 10 hours? After 20 hours? After 40 hours?

5. At a certain factor, the total cost of manufacturing q units during the daily production run is

$$C(q)=0.2q^{2}+q+900$$

dollars. From experience, it has been determined that approximately

$$q(t)=t^{2}+100t$$

units are manufactured during the first t hours of a production run. Compute the rate at which the total manufacturing cost is changing with respect to time one hour after production begins.

vi) *Derivative of implicit function*

Whether y is expressed explicitly or implicitly in terms of x, we can still differentiate to find the derivative dy/dx. If y is expressed **explicitly** in terms of x, then dy/dx will also be expressed explicitly in terms of x. If y is expressed **implicitly** in terms of x, then dy/dx will be expressed **in terms of x and y**.

Fortunately, there is a simple technique based on the chain rule that allows us to find dy/dx without first solving the equation for y explicitly. This technique is known as **implicit differentiation.** It consists differentiation of the both sides of the equation with respect to x and then solving the resultant equation algebraically for dy/dx.

Example 2.4.7: [Implicit Differentiation]: Differentiate the implicit equation $x^2y+2y^3=3x+2y$.

Solution: The implicit equation is

$$x^2y+2y^3=3x+2y. \quad (29)$$

The implicit differentiation of (29) is obtained by differentiating both sides w.r.t x:

$$\frac{d}{dx}\left(x^2y+2y^3\right)=\frac{d}{dx}(3x+2y)$$

$$\frac{d}{dx}(x^2y)+\frac{d}{dx}(2y^3)=\frac{d}{dx}(3x)+\frac{d}{dx}(2y)$$

$$2xy+x^2\frac{dy}{dx}+6y^2\frac{dy}{dx}=3+2\frac{dy}{dx}$$

$$(x^2+6y^2-2)\frac{dy}{dx}=3-2xy$$

$$\frac{dy}{dx}=\frac{3-2xy}{x^2+6y^2-2}$$

Example 2.4.8: [Implicit Differentiation]: Find the slope of a tangent line to the circle $x^2+y^2=5x+4y$ at a particular point P(5,4).

Solution: The slope of a tangent line to the given curve is *dy/dx* that can be found by taking the derivative of $x^2+y^2=5x+4y$ with respect to x:

$$\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}(5x+4y)$$

$$2x+2y\frac{dy}{dx}=5+4\frac{dy}{dx}$$

$$(2y-4)\frac{dy}{dx}=5-2x$$

$$\frac{dy}{dx}=\frac{5-2x}{2y-4}$$

At a point P(5, 4), the slope of the tangent line is:

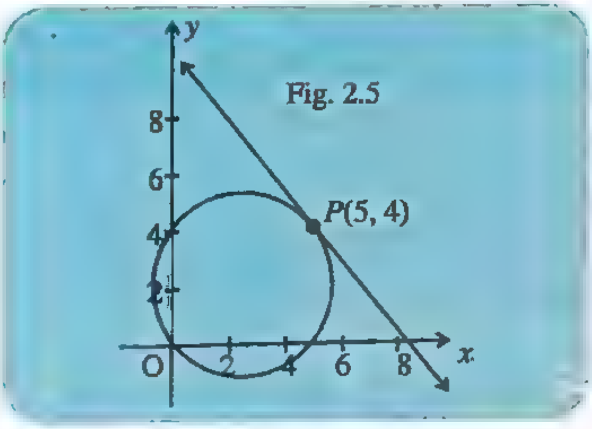


Fig. 2.5

$$\frac{dy}{dx}=\frac{5-2x}{2y-4}=\frac{5-2(5)}{2(4)-4}=\frac{-5}{4}$$

Note that the expression is undefined at $y = 2$. This makes sense, when you see that the tangent is vertical there. Look at the graph and see if you should exclude any other values.

Exercise 2.5

1. Use implicit differentiation to perform dy/dx for the following functions:

a. $x^2+y^2=25$ b. $xy=25$ c. $xy(2x+3y)=2$

d. $x^2+3xy+y^2=15$ e. $(x+y)^3+3y=3$ f. $\frac{1}{y}+\frac{1}{x}=1$

2. Arrange the following functions explicitly and implicitly to perform dy/dx:

a. $x^2y^3+y^3=12$ b. $xy+2y=x^2$

c. $x+\frac{1}{y}=5$ d. $xy-x=y+2$

3. Let $\frac{u^2}{a^2}+\frac{v^2}{b^2}=1$, where a and b are nonzero constants. Find:

a. $\frac{du}{dv}$ b. $\frac{dv}{du}$

4. Let $(a-b)u^3-(a+b)v^2=c$, where a, b and c are constants. Find:

a. $\frac{du}{dv}$ b. $\frac{dv}{du}$

5. Determine the slope of the tangent line to the curve $3x^2-7y^2+14y=27$ at the point $P(-3,0)$.

6. The graph of $x^3y^3+4y=3x^2$ is a curve that passes through the point P(2, 1) What is the slope of the curve at that point?

7. In biophysics, the equation $(L+m)(V+n)=k$ is called the fundamental equation of **muscle contraction**, where m, n and k are constants, and V is the velocity of the shortening of muscle fibers for a muscle subjected to a load of L. Find dL/dV using implicit differentiation.

2.5 Differentiation of Trigonometric and Inverse Trigonometric Functions

- ***Differentiation of trigonometric functions***

Derivative of sinx: If $y=\sin x$, then the derivative of $y=\sin x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x},\quad y=f(x)=\sin x$$

$$=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sin x\cos\Delta x+\cos x\sin\Delta x-\sin x}{\Delta x},$$

$$=\lim_{\Delta x\to 0}\left[\sin x\left(\frac{\cos\Delta x}{\Delta x}\right)+\cos x\left(\frac{\sin\Delta x}{\Delta x}\right)-\frac{\sin x}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}\left[\sin x\left(\frac{\cos\Delta x}{\Delta x}\right)-\frac{\sin x}{\Delta x}+\cos x\left(\frac{\sin\Delta x}{\Delta x}\right)\right]$$

$$= \sin x\lim_{\Delta x\to 0}\left(\frac{\cos\Delta x-1}{\Delta x}\right)+\cos x\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}$$

$$= \sin x(0)+\cos x(1)=\cos x,$$

Derivative of cos x: If $y = \cos x$, then the derivative of $y = \cos x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x},\quad y=f(x)=\cos x \qquad (30)$$

$$=\lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x}$$

The expression $\cos(x+\Delta x)-\cos x$ is replaced by

$$\cos(x+\Delta x)-\cos x=-2\sin\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right) \qquad (31)$$

Introduce $A+B=x+\Delta x$ and $A-B=x$. The system is solved for the unknowns A and B to obtain:

$$A=\frac{2x+\Delta x}{2}=x+\frac{\Delta x}{2},\ B=\frac{\Delta x}{2}$$

Equation (31) is used in equation (30) to obtain:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{-2\sin\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$$

$$=-\lim_{\Delta x\to 0}\sin\left(x+\frac{\Delta x}{2}\right)\lim_{\Delta x\to 0}\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$$

$$=-\sin(x+0)(1)=-\sin x,$$

Derivative of tan x: If $y = \tan x$, then the derivative of $y = \tan x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}, \quad y=f(x)=\tan x$$

$$=\lim_{\Delta x\to 0}\frac{\tan(x+\Delta x)-\tan x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\frac{\sin(x+\Delta x)}{\cos(x+\Delta x)}-\frac{\sin x}{\cos x}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)\cos x-\sin x\cos(x+\Delta x)}{\Delta x\cos x\cos(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x-x)}{\Delta x\cos x\cos(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\cos(x+\Delta x)}\frac{1}{\cos x}\frac{\sin\Delta x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\cos(x+\Delta x)}\lim_{\Delta x\to 0}\frac{1}{\cos x}\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}$$

$$=\frac{1}{\cos x}\frac{1}{\cos x}(1)$$

$$=\frac{1}{\cos^2 x}=\sec^2 x,$$

Derivative of sec x: If $y=\sec x$, then the derivative of $y=\sec x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}, \quad y=f(x)=\sec x$$

$$=\lim_{\Delta x\to 0}\frac{\sec(x+\Delta x)-\sec x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\frac{1}{\cos(x+\Delta x)}-\frac{1}{\cos x}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\cos x-\cos(x+\Delta x)}{\Delta x\cos x\cos(x+\Delta x)}=\lim_{\Delta x\to 0}\frac{2\sin\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x\cos x\cos(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \sin\left(x+\frac{\Delta x}{2}\right) \frac{1}{\cos(x+\Delta x)} \frac{1}{\cos x} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$$

$$= \lim_{\Delta x \to 0} \sin\left(x+\frac{\Delta x}{2}\right) \lim_{\Delta x \to 0} \frac{1}{\cos x} \lim_{\Delta x \to 0} \frac{1}{\cos(x+\Delta x)} \lim_{\Delta x \to 0} \frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$$

$$= \sin x \frac{1}{\cos x} \frac{1}{\cos x}(1) = \frac{1}{\cos x} \frac{\sin x}{\cos x} = \sec x \tan x$$

Derivative of cosec x: If y = cosec x, then the derivative of y =cosec x by first principle rule is:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x) = \operatorname{cosec} x$$

$$= \lim_{\Delta x \to 0} \frac{\operatorname{cosec}(x+\Delta x) - \operatorname{cosec} x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{\sin(x+\Delta x)} - \frac{1}{\sin x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x - \sin(x+\Delta x)}{\Delta x \sin x \sin(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x - \sin x \cos \Delta x - \cos x \sin \Delta x}{\Delta x \sin x \sin(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{-\sin x(\cos \Delta x - 1) - \cos x \sin \Delta x}{\Delta x \sin x \sin(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{-\sin x(\cos \Delta x - 1)}{\Delta x \sin x \sin(x+\Delta x)} - \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x}{\Delta x \sin x \sin(x+\Delta x)}$$

$$= -\lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} \lim_{\Delta x \to 0} \frac{1}{\sin(x+\Delta x)} - \frac{\cos x}{\sin x} \lim_{\Delta x \to 0} \frac{1}{\sin(x+\Delta x)} \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}$$

$$= (0)\left(\frac{1}{\sin x}\right) - \cot x\left(\frac{1}{\sin x}\right)(1) = -\cot x \operatorname{cosec} x,$$

Derivative of cot x: If $y = \cot x$, then the derivative of $y = \cot x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x},\quad y=f(x)=\cot x$$

$$=\lim_{\Delta x\to 0}\frac{\cot(x+\Delta x)-\cot x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\frac{\cos(x+\Delta x)}{\sin(x+\Delta x)}-\frac{\cos x}{\sin x}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sin x\cos(x+\Delta x)-\cos x\sin(x+\Delta x)}{\Delta x\sin x\sin(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{\sin(x-x-\Delta x)}{\Delta x\sin x\sin(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\sin(x+\Delta x)}\frac{1}{\sin x}\frac{-\sin\Delta x}{\Delta x},\qquad \sin(-\Delta x)=-\sin\Delta x$$

$$=-\lim_{\Delta x\to 0}\frac{1}{\sin(x+\Delta x)}\lim_{\Delta x\to 0}\frac{1}{\sin x}\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}$$

$$=-\frac{1}{\sin x}\frac{1}{\sin x}(1)=-\frac{1}{\sin^2 x}=-\operatorname{cosec}^2 x,$$

These trigonometric formulas are listed in the box:

1. $\frac{d}{dx}(\sin x)=\cos x$
2. $\frac{d}{dx}(\cos x)=-\sin x$
3. $\frac{d}{dx}(\tan x)=\sec^2 x$
4. $\frac{d}{dx}(\operatorname{cosec} x)=-\cot x\operatorname{cosec} x$
5. $\frac{d}{dx}(\sec x)=\tan x\sec x$
6. $\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the trigonometric functions, as summarized in the box:

1. $\frac{d}{dx}(\sin u)=\cos u\frac{d}{dx}(u)$
2. $\frac{d}{dx}(\cos u)=-\sin u\frac{d}{dx}(u)$
3. $\frac{d}{dx}(\tan u)=\sec^2 u\frac{d}{dx}(u)$
4. $\frac{d}{dx}(\operatorname{cosec} u)=-\cot u\operatorname{cosec} u\frac{d}{dx}(u)$
5. $\frac{d}{dx}(\sec u)=\tan u\sec u\frac{d}{dx}(u)$
6. $\frac{d}{dx}(\cot u)=-\operatorname{cosec}^2 u\frac{d}{dx}(u)$

Example 2.5.1:[Trigonometric Function]:Differentiate the following trigonometric functions:

a. $p(t)=(t^2+t)\sin t$ b. $f(x)=\dfrac{1+\sin x}{2-\cos x}$

Solution:

a. If the given function is $p(t)=uv=(t^2+t)\sin t$, then the product rule of differentiation w.r.t t is used to obtain:

$$\frac{dp}{dt}=u\frac{dv}{dt}+v\frac{du}{dt}$$

$$=(t^2+t)\frac{d}{dt}(\sin t)+(\sin t)\frac{d}{dt}(t^2+t)=(t^2+t)\cos t+\sin t(2t+1)$$

b. If the given function is $f(x)=\dfrac{u}{v}=\dfrac{1+\sin x}{2-\cos x}$, then the quotient rule of differentiation w.r.t x is used to obtain :

$$\frac{df}{dx}=\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{\frac{du}{dx}v-\frac{dv}{dx}u}{v^2}$$

$$=\frac{(\cos x)(2-\cos x)-(\sin x)(1+\sin x)}{(2-\cos x)^2}$$

$$=\frac{2\cos x-\cos^2x-\sin x-\sin^2x}{(2-\cos x)^2}$$

$$=\frac{2\cos x-\sin x-(\cos^2x+\sin^2x)}{(2-\cos x)^2}=\frac{2\cos x-\sin x-1}{(2-\cos x)^2}, \sin^2x+\cos^2x=1$$

Example 2.5.2: [Trigonometric Function]: Differentiate the following trigonometric functions:

a. $f(x)=\sec x\tan x$ b. $f(x)=\dfrac{x^2+\tan x}{3x+2\tan x}$

Solution:

a. If the given function is $f(x)=uv=\sec x\tan x$, then the product rule of differentiation w.r.t x is used to obtain:

$$\frac{df}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

$$=\sec x\frac{d}{dx}(\tan x)+(\tan x)\frac{d}{dx}(\sec x)$$

$$=\sec x(\sec^2 x)+\tan x(\sec x\tan x)=\sec^3 x+\sec x\tan^2 x$$

b. If the given function is $f(x)=\dfrac{x^2+\tan x}{3x+2\tan x}$, then the quotient rule of differentiation w.r.t x is used to obtain :

$$\frac{df}{dx}=\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{\frac{du}{dx}v-\frac{dv}{dx}u}{v^2}$$

$$=\frac{(2x+\sec^2 x)(3x+2\tan x)-(3+2\sec^2 x)(x^2+\tan x)}{(3x+2\tan x)^2}$$

$$=\frac{3x^2+(4x-3)\tan x+x(3-2x)\sec^2 x}{(3x+2\tan x)^2}$$

- ***Differentiation of inverse trigonometric functions***

Derivative of $\sin^{-1}x$**:** If $y=\sin^{-1}x$, then $x=\sin y$. The differentiation of x=sin y w.r.t y is:

$$\frac{dx}{dy}=\cos y$$

Take its reciprocal to obtain the derivative of y w.r.t x::

$$\frac{dy}{dx}=\frac{1}{\cos y}$$

$$=\pm\frac{1}{\sqrt{1-\sin^2 y}}=\pm\frac{1}{\sqrt{1-x^2}},\quad \sin^2 y+\cos^2 y=1,\ \sin y=x$$

Here, the sign of the radical is the same as that of cos y. By definition of $\sin^{-1}x$:

$$-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}\quad or\quad -\frac{\pi}{2}\le y\le\frac{\pi}{2},$$

Hence, cos y is positive:

$$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$$

Derivative of $\cos^{-1}x$: If $y=\cos^{-1}x$, then $x=\cos y$. The differentiation of x= cos y w.r.t y is:

$$\frac{dx}{dy}=-\sin y$$

Take its reciprocal to obtain the derivative of y w.r.t x:

$$\frac{dy}{dx}=-\frac{1}{\sin y}$$

$$=\pm\frac{-1}{\sqrt{1-\cos^2 y}}=\pm\frac{-1}{\sqrt{1-x^2}},\quad \sin^2 y+\cos^2 y=1, \cos y=x$$

Here, the sign of the radical is the same as that of sin y. By definition of $\cos^{-1}x$:

$$0\le\cos^{-1}x\le\pi \quad or \quad 0\le y\le\pi,$$

Also, if y lies between 0 and π, then, sin y is necessarily positive. Hence

$$\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}$$

Derivative of $\tan^{-1}x$: If $y=\tan^{-1}x$, then $x=\tan y$. The differentiation of x=tan y w.r.t y is:

$$\frac{dx}{dy}=\sec^2 y$$

Take its reciprocal to obtain the derivative of y w.r.t x:

$$\frac{dy}{dx}=\frac{1}{\sec^2 y}$$

$$=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2},\quad \sec^2 y=1+\tan^2 y, \tan y=x$$

Derivative of $\sec^{-1}x$: If $y=\sec^{-1}x$, then $x=\sec y$. The differentiation of x=sec y w.r.t y is:

$$\frac{dx}{dy} = sec y \tan y$$

Take its reciprocal to obtain the derivative of y w.r.t x:

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$= \pm \frac{1}{sec y\sqrt{\sec^2 y - 1}} = \pm \frac{1}{x\sqrt{x^2 - 1}}, 1 + \tan^2 y = \sec^2 y, \sec y = x$$

We take + sign before the radical sign to obtain:

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$$

Derivative of $cosec^{-1}x$ **:** If $y = cosec^{-1}x$, then $x = cosec y$. The differentiation of x=cosec y w.r.t y is.

$$\frac{dx}{dy} = -cosec y \cot y$$

Take its reciprocal to obtain the derivative of y w.r.t x:

$$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec} y \cot y}$$

$$= \pm \frac{-1}{cosec y\sqrt{\operatorname{cosec}^2 y - 1}}, \quad 1 + \cot^2 y = \operatorname{cosec}^2 y$$

$$= \pm \frac{-1}{x\sqrt{x^2 - 1}}, \qquad \operatorname{cosec} y = x$$

We take + sign before the radical sign to obtain:

$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2 - 1}}$$

Derivative of $\cot^{-1} x$ **:** If $y = \cot^{-1} x$, then $x = \cot y$. The differentiation of **x=cot y** w.r.t y is:

$$\frac{dx}{dy} = -cosec^2 y$$

Take its reciprocal to obtain the derivative of y w.r.t x:

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec}^2 y} = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}, \quad \operatorname{cosec}^2 y = 1+\cot^2 y, \ \cot y = x$$

These inverse trigonometric formulas are listed in the box:

1. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$
4. $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$
5. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$
6. $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the inverse trigonometric functions, as summarized in the box:

1. $\frac{d}{dx}(\sin^{-1}u) = \frac{1}{\sqrt{1-u^2}}\frac{d}{dx}(u)$
2. $\frac{d}{dx}(\cos^{-1}u) = \frac{-1}{\sqrt{1-u^2}}\frac{d}{dx}(u)$
3. $\frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2}\frac{d}{dx}(u)$
4. $\frac{d}{dx}(\operatorname{cosec}^{-1}u) = \frac{-1}{u\sqrt{u^2-1}}\frac{d}{dx}(u)$
5. $\frac{d}{dx}(\sec^{-1}u) = \frac{1}{u\sqrt{u^2-1}}\frac{d}{dx}(u)$
6. $\frac{d}{dx}(\cot^{-1}u) = \frac{-1}{1+u^2}\frac{d}{dx}(u)$

Example 2.5.3: [Inverse Trigonometric Function]: Differentiate the following inverse trigonometric functions:

a. $y = \tan^{-1}\sqrt{x}$ b. $f(x) = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$

Solution:

a. If $y = f(x) = \tan^{-1}\sqrt{x} = f(u) = \tan^{-1}u$ with $u = \sqrt{x}$, then the chain rule of differentiation w.r.t x is used to obtain :

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$

$$=\frac{d}{du}(\tan^{-1}u)\frac{d}{dx}(\sqrt{x})$$

$$=\left(\frac{1}{1+u^2}\right)\left(\frac{1}{2}x^{-1/2}\right)$$

$$=\frac{1}{2\sqrt{x}(1+u^2)}=\frac{1}{2\sqrt{x}(1+x)},\quad u=\sqrt{x}$$

b. If $f(x)=\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$ with $u=\frac{x-x^{-1}}{x+x^{-1}}=\frac{x^2-1}{x^2+1}$, then the chain rule of differentiation w.r.t x is used to obtain :

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$

$$=\frac{d}{dx}(\cos^{-1}u)\frac{d}{dx}(u),\quad u=\frac{x-x^{-1}}{x+x^{-1}}=\frac{x^2-1}{x^2+1}$$

$$=\frac{-1}{\sqrt{1-u^2}}\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$$

$$=\frac{-1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\left[\frac{(2x)(x^2+1)-(x^2-1)(2x)}{(x^2+1)^2}\right]$$

$$=\frac{-1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\left[\frac{4x}{(x^2+1)^2}\right]=\frac{-2}{x^2+1}$$

Exercise 2.6

1. Use any suitable rule of differentiation to perform $\frac{dy}{dx}$ for the following functions:

a. $y=\sqrt{\frac{1-\cos x}{1+\cos x}}$ b. $y=\cos\left(x+\frac{\pi}{2}\right)$ c. $y=\sin(\sin x)$

d. $y = \sin x \cos x$ e. $y = \dfrac{\sin x}{\cos x}$ f. $y = \sin^3(\pi x^2)$

2. Use any suitable rule of differentiation to perform $\dfrac{dy}{dx}$ for the following functions:

a. $y = 2\cot 3x$ b. $y = \sec \pi x$ c. $y = 4\operatorname{cosec} 2x$

d. $y = 2\tan(x+3)^2$ e. $y = 4\cot\sqrt{x^2-1}$ f. $y = \sec^2 x^3$

g. $y = 2\operatorname{cosec}^3(x+2)$ h. $y = \dfrac{1+\tan 2x}{\operatorname{cosec} 3x}$

3. Use any suitable rule of differentiation to perform $\dfrac{dy}{dx}$ for the following functions:

a. $y = \cos^{-1}(x+4)$ b. $y = \tan^{-1}(11x)$

c. $y = \left(\sin^{-1}x\right)^2$ d. $y = x^3\sin^{-1}2x$

e. $y = \operatorname{cosec}^{-1}(x+3)$ f. $y = (1+\cot^{-1}3x)^3$

4. Suppose profits on the sale of swimming suits in a departmental store are given approximately by

$$P(t) = 5 - 5\cos\frac{\pi t}{26}, \quad 0 \le t \le 104$$

where P(t) is profit (in hundreds of dollars) for a week of sales t weeks after January first.

a. What is the rate of change of profit t weeks after the first of the year?

b. What is the rate of change of profit 8 weeks after the first of the year? 26 weeks after the first of the year? 50 weeks after the first of the year?

5. A normal seated adult breathes in and exhales about 0.8 liter of air every 4 seconds. The volume of air V (t) in the lungs t seconds after exhaling is given approximately by

$$V(t) = 0.45 - 0.35\cos\frac{\pi t}{2}, \quad 0 \le t \le 8$$

a. What is the rate of flow of air t seconds after exhaling?

b. What is the rate of flow of air 3 seconds after exhaling? 4 seconds after exhaling? 5 seconds after exhaling?

2.6 Differentiation of Exponential and Logarithmic Functions

The goal of this section is to develop the differential calculus of logarithmic and exponential functions. We shall begin by deriving differentiation formulas for $\ln x$ and e^x. The derived formulas will be applied to a number of differentiation problems and applications.

i) *Derivative of e^x and a^x first principle rule*

Derivative of e^x : If $y = e^x$, then the derivative of $y = e^x$ by first principle rule is:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x) = e^x$$

$$= \lim_{\Delta x \to 0} \frac{e^{(x+\Delta x)} - e^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^x e^{\Delta x} - e^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^x (e^{\Delta x} - 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x(1) = e^x, \quad \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1$$

Derivative of a^x : If $y = a^x$, then the derivative of $y = a^x$ by first principle rule is:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x) = a^x$$

$$= \lim_{\Delta x \to 0} \frac{a^{(x+\Delta x)} - a^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{a^x a^{\Delta x} - a^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{a^x (a^{\Delta x} - 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

$$= a^x \log_e a, \quad \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = \log_e a = \ln a$$

ii) *Derivative of lnx and $\log_a x$ by first principle rule*

Derivative of $\ln x$: If $y = \ln x$, then the derivative of $y = \ln x$ by first principle rule is:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x) = \ln x \qquad (32)$$

$$= \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \ln\left(\frac{x+\Delta x}{x}\right), \quad \textit{Lograithmic} - \textit{rule}$$

$$= \lim_{\Delta x \to 0} \frac{x}{x}\frac{1}{\Delta x} \ln\left(1+\frac{\Delta x}{x}\right) \quad \textit{multipy and divide out by } x$$

$$= \lim_{\Delta x \to 0} \frac{1}{x}\left(\frac{x}{\Delta x}\right) \ln\left(1+\frac{\Delta x}{x}\right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{x} \ln\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}, \quad \textit{Logrithmic rule}$$

Let $s = \Delta x/x$. For x fixed, if $\Delta x \to 0$, then $s \to 0$. Equation (32) with these substitutions becomes:

$$\frac{dy}{dx} = \frac{1}{x} \lim_{\Delta x \to 0} \ln\left(1+\frac{\Delta x}{x}\right)^{x/\Delta x}$$

$$= \frac{1}{x} \lim_{s \to 0} \ln(1+s)^{1/s} \qquad , \lim_{s \to 0}(1+s)^{1/s} = e, \ \ln e = \log_e e = 1$$

$$= \frac{1}{x} \ln \lim_{s \to 0}(1+s)^{1/s}$$

$$= \frac{1}{x} \ln e = \frac{1}{x}$$

Derivative of $\log_a x$: If $y = \log_a x$, then the derivative of $y = \log_a x$ by first principle rule is:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x) = \log_a x \qquad (33)$$

$$= \lim_{\Delta x \to 0} \frac{\log_a(x+\Delta x) - \log_a x}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \log_a\left(\frac{x+\Delta x}{x}\right),$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right)$$

$$= \lim_{\Delta x \to 0} \frac{x}{x} \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right), \quad \textit{multiply and divide out by } x$$

$$= \frac{1}{x} \lim_{\Delta x \to 0} \frac{x}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right)$$

$$= \frac{1}{x} \lim_{\Delta x \to 0} \log_a \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}, \textit{Logrithmic rule}$$

Let $s = \Delta x/x$. For x fixed, if $\Delta x \to 0$, then $s \to 0$. Equation (33) with these substitutions becomes:

$$\frac{dy}{dx} = \frac{1}{x} \lim_{\Delta x \to 0} \log_a \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}, \qquad \lim_{s \to 0} (1+s)^{1/s} = e$$

$$= \frac{1}{x} \lim_{s \to 0} \log_a (1+s)^{1/s}$$

$$= \frac{1}{x} \log_a \lim_{s \to 0} (1+s)^{1/s} = \frac{1}{x} \log_a e$$

These exponential and logarithmic formulas are listed in the box:

1. $\frac{d}{dx}(e^x) = e^x$
2. $\frac{d}{dx}(a^x) = a^x \log_e a$
3. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
4. $\frac{d}{dx}(\log_a x) = \frac{1}{x} \log_a e = \frac{1}{x \log_e a} = \frac{1}{x \ln a}$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the exponential and logarithmic functions, as summarized in the box:

1. $\frac{d}{dx}(e^u) = e^u \frac{d}{dx}(u)$
2. $\frac{d}{dx}(a^u) = a^u \log_e a \frac{d}{dx}(u)$
3. $\frac{d}{dx}(\ln u) = \frac{1}{u} \frac{d}{dx}(u)$
4. $\frac{d}{dx}(\log_a u) = \frac{1}{u} \log_a e \frac{d}{dx}(u)$

Example 2.6.1: [Exponential & Logarithmic Functions]: Differentiate the following functions:

a. $f(x)=7^{(4-3x^5)}$

b. $f(x)=\log_{10}\sqrt{x^2-7x}+x^3$

c. $f(x)=\ln(e^{mx}+e^{-mx})$

d. $f(x)=\dfrac{e^{3x}}{\ln x}$

Solution:

a. If the given function is $f(x)=7^{(4-3x^5)}$, then the derivative of a given function w.r.t x is:

$$\frac{df}{dx}=\frac{d}{dx}\left[7^{(4-3x^5)}\right]$$

$$=\frac{d}{du}(7^u)\frac{d}{dx}(u)$$

$$=7^u\log_e 7\frac{d}{dx}(4-3x^5),\quad u=(4-3x^5)$$

$$=7^{(4-3x^5)}\log_e 7(-15x^4)$$

$$=-15x^4 7^{(4-3x^5)}\log_e 7$$

b. If the given function is $f(x)=\log_{10}\sqrt{(x^2-7x)}+x^3$, then the derivative of a given function w.r.t x is:

$$\frac{df}{dx}=\frac{d}{dx}\left[\log_{10}\sqrt{(x^2-7x)}+x^3\right]$$

$$=\frac{d}{dx}[\log_{10}\sqrt{(x^2-7x)}]+\frac{d}{dx}(x^3)$$

$$=\frac{1}{u}\log_{10}e\frac{d}{dx}(u)+3x^2,\quad u=(x^2-7x)$$

$$=\frac{1}{x^2-7x}\log_{10}e\frac{d}{dx}(x^2-7x)+3x^2$$

$$=\frac{2x-7}{x^2-7x}\log_{10}e+3x^2$$

c. If the given function is $f(x)=\ln(e^{mx}+e^{-mx})$, then the derivative of a given function w.r.t x is:

$$\frac{df}{dx}=\frac{d}{dx}[\ln(e^{mx}+e^{-mx})]$$
$$=\frac{d}{du}(\ln u)\frac{d}{dx}(e^{mx}+e^{-mx}),\quad u=e^{mx}+e^{-mx}$$
$$=\frac{1}{\ln u}(me^{mx}-me^{-mx})$$
$$=\frac{1}{\ln(e^{mx}+e^{-mx})}(m)(e^{mx}-e^{-mx})$$
$$=\frac{m(e^{mx}-e^{-mx})}{\ln(e^{mx}+e^{-mx})}$$

d. If the given function is $f(x)=e^{2x}/\ln x$, then the derivative of a given function w.r.t x is:

$$\frac{df}{dx}=\frac{d}{dx}\left(\frac{e^{2x}}{\ln x}\right)$$
$$=\frac{2e^{2x}\ln x-\frac{1}{x}e^{2x}}{(\ln x)^2}$$
$$=\frac{2xe^{2x}\ln x-e^{2x}}{x(\ln x)^2}=\frac{e^{2x}(2x\ln x-1)}{x(\ln x)^2}$$

iii) *Use of logarithmic differentiation to algebraic expressions involving product, quotient and power*

Logarithmic differentiation is a procedure in which logarithms are used to trade the task of differentiating products and quotients for that of differentiating sums and differences. It is especially valuable as a means for handling complicated product or quotient functions and power functions where variables appear in both the base and the exponent.

Example 2.6.2: [Logarithmic Functions]: Differentiate the following functions

a. $y=\ln\left[\frac{x(x^2-3)^2}{(x^2-4)^{\frac{1}{2}}}\right]$ b. $y=x^{\sin x}$

Solution: In examining the function, we see that inside the brackets, we have a product, a quotient, and a power. Thus, to find the derivative, we would apply the rules for finding the derivative of lnu and then find the derivatives of a product, quotient and power. This all seems rather complicated, but can be simplified, if we use the rules of logarithms and write the function in the following manner:

a. If the given function after simplification is

$$y = \ln\left[\frac{x(x^2-3)^2}{(x^2-4)^{\frac{1}{2}}}\right]$$

$$= \ln[x(x^2-3)^2] - \ln[(x^2-4)^{\frac{1}{2}}], \quad \textit{logarithms rules}$$

$$= \ln x + \ln(x^2-3)^2 - \ln(x^2-4)^{\frac{1}{2}}$$

$$= \ln x + 2\ln(x^2-3) - \frac{1}{2}\ln(x^2-4),$$

then the derivative of y w.r.t x is

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln x + 2\ln(x^2-3) - \frac{1}{2}\ln\left(x^2-4\right)\right]$$

$$= \frac{1}{x} + 2\frac{1}{x^2-3}(2x) - \frac{1}{2}\cdot\frac{1}{x^2-4}(2x) = \frac{4x^4-20x^2+12}{x(x^2-3)(x^2-4)}$$

b. If the given function $y = x^{\sin x}$ after simplification is:

$\ln y = \ln(x^{\sin x})$, *Taking* ln *of both sides*

$\ln y = \sin x \ \ln x$

then on differentiation w.r.t x, it becomes:

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}(\sin x\ln x)$$

$$\frac{1}{y}\frac{dy}{dx}=\sin x\frac{d}{dx}(\ln x)+\ln x\frac{d}{dx}(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{\sin x}{x}+\ln x\cos x$$

$$\frac{dy}{dx}=y\left[\frac{\sin x}{x}+\ln x\cos x\right]=x^{\sin x}\left[\frac{\sin x}{x}+\ln x\cos x\right]$$

2.7 Differentiation of Hyperbolic and Inverse Hyperbolic Functions

The concept of hyperbolic functions is completely cleared in previous section. The differentiation of hyperbolic functions can be found as follows:

i) *Differentiation of hyperbolic functions*

Derivative of $\sin hx$:If $y=\sin hx=\frac{e^x-e^{-x}}{2}$, then on differentiation w.r.t x, it becomes

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{e^x-e^{-x}}{2}\right]$$

$$=\frac{1}{2}\left[\frac{d}{dx}(e^x)-\frac{d}{dx}(e^{-x})\right]=\frac{1}{2}(e^x+e^{-x})=\cosh x$$

Derivative of $\cos hx$: , If $y=\cos hx=\frac{e^x+e^{-x}}{2}$, then on differentiation w.r.t x, it becomes

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{e^x+e^{-x}}{2}\right]$$

$$=\frac{1}{2}\left[\frac{d}{dx}(e^x)+\frac{d}{dx}(e^{-x})\right]=\frac{1}{2}(e^x-e^{-x})=\sinh x$$

Derivative of $\tan hx$ **:** If $y=\tan hx=\dfrac{\sin hx}{\cos hx}$, then on differentiation w.r.t x through quotient rule, it becomes

$$\frac{dy}{dx}=\frac{\cos hx\frac{d}{dx}(\sin hx)-\sin hx\frac{d}{dx}(\cos hx)}{\cosh^2 x}, \qquad \cosh^2 x-\sinh^2 x=1$$

$$=\frac{\cos hx\cos hx-\sin hx sinhx}{\cosh^2 x}$$

$$=\frac{\cosh^2 x-\sinh^2 x}{\cosh^2 x}=\frac{1}{\cosh^2 x}=\sec h^2 x$$

Derivative of $\sec hx$ **:** If $y=\sec hx=\dfrac{1}{\cos hx}$, then on differentiation w.r.t x through quotient rule, it becomes

$$\frac{dy}{dx}=\frac{\cos hx\frac{d}{dx}(1)-(1)\frac{d}{dx}(\cos hx)}{\cosh^2 x}$$

$$=\frac{\cos hx(0)-\sin hx}{\cosh^2 x}$$

$$=\frac{-\sinh x}{\cosh^2 x}=-\frac{\sin hx}{\cos hx}\,\frac{1}{\cos hx}=-tanhx\sec hx$$

Derivative of $\operatorname{cosec} hx$ **:** If $y=\operatorname{cosec} hx=\dfrac{1}{\sin hx}$, then on differentiation w.r.t x through quotient rule, it becomes

$$\frac{dy}{dx}=\frac{\sinh x\frac{d}{dx}(1)-(1)\frac{d}{dx}(\sin hx)}{\sin h^2 x}$$

$$=\frac{\sin hx(0)-\cos hx}{\sin h^2 x}$$

$$=\frac{-\cos hx}{\sinh^2 x}$$

$$= -\frac{\cos hx}{\sin hx}\,\frac{1}{\sin hx} = -cothx\ \ cosec\,hx$$

Derivative of $\cot hx$**:** If $y = \cot hx = \frac{\cos hx}{\sin hx}$, then on differentiation w.r.t x through quotient rule, it becomes

$$\frac{dy}{dx} = \frac{\sin hx\frac{d}{dx}(\cos hx) - \cos hx\frac{d}{dx}(\sin hx)}{\sinh^2 x},\quad \cosh^2 x - \sin h^2 x = 1$$

$$= \frac{\sin hx \sin hx - \cos hx \cos hx}{\sinh^2 x}$$

$$= \frac{-(\cosh^2 x - \sinh^2 x)}{\sinh^2 x} = -\frac{1}{\sinh^2 x} = -cosech^2 x$$

These hyperbolic formulas are listed in the box:

$1.\ \frac{d}{dx}(\sinh x) = \cosh x$	$4.\ \frac{d}{dx}(cosechx) = -cothx\ cosechx$
$2.\ \frac{d}{dx}(\cosh x) = \sin hx$	$5.\ \frac{d}{dx}(sechx) = -\tan hx \sec hx$
$3.\ \frac{d}{dx}(\tan hx) = \sec h^2 x$	$6.\ \frac{d}{dx}(\cot x) - \mathrm{cosec}^2 x$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the hyperbolic functions, as summarized in the box:

$1.\ \frac{d}{dx}(\sinh u) = \cosh u\frac{d}{dx}(u)$	$4.\ \frac{d}{dx}(cosechu) = -\coth u\ cosechu\frac{d}{dx}(u)$
$2.\ \frac{d}{dx}(\cosh u) = \sin hu\frac{d}{dx}(u)$	$5.\ \frac{d}{dx}(sechu) = -\tan hu \sec hu\frac{d}{dx}(u)$
$3.\ \frac{d}{dx}(\tan hu) = \sec h^2 u\frac{d}{dx}(u)$	$6.\ \frac{d}{dx}(\coth x) = -\mathrm{cosec}^2 u\frac{d}{dx}(u)$

Example 2.7.1: [Hyperbolic Function]: Differentiate the following functions:

a. $y = \cos h(1-2x^2)$ b. $y = \sec h\left(\frac{1-x}{1+x}\right)$

Solution:

a. If the given function is $y = \cos h(1-2x^2)$, then the derivative of y w.r.t x is:

$$y' = \frac{d}{dx}[\cos h(1-2x^2)]$$

$$= \frac{d}{du}(\cos hu)\frac{d}{dx}(1-2x^2),\ u = 1-2x^2 = \sin hu(-4x) = -4x\cos h(1-2x^2)$$

b. If the given function is $y = sech\left[\frac{1-x}{1+x}\right]$, then the derivative of y w.r.t x is:

$$y' = \frac{d}{dx}\left[\sec h\left(\frac{1-x}{1+x}\right)\right]$$

$$= \frac{d}{du}(sechu)\frac{d}{dx}(u) = -tanhu\sec hu\frac{d}{dx}\left(\frac{1-x}{1+x}\right),\quad u = \frac{1-x}{1+x}$$

$$= -\tanh u\ sechu\left[\frac{(-1)(1+x)-(1-x)(1)}{(1+x)^2}\right]$$

$$= \frac{2}{(1+x)^2}\tanh\left(\frac{1-x}{1+x}\right)\sec h\left(\frac{1-x}{1+x}\right)$$

ii) *Differentiation of inverse hyperbolic functions*

Derivative of $\sinh^{-1} x$ **:** If $y = \sinh^{-1}x$, then $x = \sinh y$, the differentiation of $x = \sinh y$ w.r.t y is:

$$\frac{dx}{dy} = coshy$$

Take its reciprocal to obtain the derivative of y w.r.t x:

$$\frac{dy}{dx}=\frac{1}{\cosh y}$$

$$=\pm\frac{1}{\sqrt{1+\sinh^2 y}}=\pm\frac{1}{\sqrt{1+x^2}},\quad \sinh y=x,\quad \cosh^2 y-\sinh^2 y=1$$

Here, the sign of the radical is the same as that of coshy which we know is always positive. Hence,

$$\frac{d}{dx}(\sin h^{-1}x)=\frac{1}{\sqrt{1+x^2}}$$

Derivative of $\cosh^{-1}x$**:** If If $y=\cosh^{-1}x$, then $x=\cosh y$, then the differentiation of $x=\cosh y$ w.r.t y is:

$$\frac{dx}{dy}=\sinh y$$

Take its reciprocal to obtain the derivative of y w.r.t x:

$$\frac{dy}{dx}=\frac{1}{\sinh y}$$

$$=\pm\frac{1}{\sqrt{\cosh^2 y-1}}=\pm\frac{1}{\sqrt{x^2-1}},\quad \cosh y=x,\quad \cosh^2 y-\sinh^2 y=1$$

Here, the sign of the radical is the same as that of coshy which we know is always positive. Hence,

$$\frac{d}{dx}(\cosh^{-1}x)=\frac{1}{\sqrt{x^2-1}}$$

Derivative of $\tanh^{-1}x$: If $y=\tanh^{-1}x$, then $x=\tanh y$, then the differentiation of $x=\tanh y$ w.r.t y is:

$$\frac{dx}{dy}=\sec h^2 y$$

Take its reciprocal to obtain the derivative of y w.r.t x:

$$\frac{dy}{dx}=\frac{1}{sech^2 y}$$

$$=\frac{1}{1-\tanh^2 y}$$

$$=\frac{1}{1-x^2},\quad sech^2 y=1-\tanh^2 y, |x|<1, \tanh y=x$$

Derivative of $\sec h^{-1}x$: If $y=\sec h^{-1}x$, then $x=\text{sech } y$. The differentiation of $x=\sec hy$ w.r.t y is:

$$\frac{dx}{dy}=-\sec hy\, tanhy$$

Take its reciprocal to obtain the derivative of y w.r.t x:

$$\frac{dy}{dx}=\frac{-1}{\sec hy \tan hy}$$

$$=\pm\frac{-1}{\sec hy\sqrt{1-\sec h^2 y}}\qquad ,\ \sec hy=x$$

$$=\pm\frac{-1}{x\sqrt{1-x^2}},\quad 1-\sec h^2 y=\tan h^2 y$$

Here, the sign of the radical is the same as that of tanhy which, but we know that $\sec h^{-1}x$ is always positive, so that tanhy is always positive. Hence,

$$\frac{d}{dx}(sech^{-1}x)=\frac{-1}{x\sqrt{1-x^2}}$$

Derivative of $\cos ech^{-1}x$: If $y=\operatorname{cosec}h^{-1}x$, then $x=\cos echy$. The differentiation of $x=\operatorname{cosec}hy$ w.r.t y is:

$$\frac{dx}{dy}=-\operatorname{cosec}hy\,cothy$$

Take its reciprocal to obtain the derivative of y w.r.t x:

$$\frac{dy}{dx}=\frac{-1}{\operatorname{cosec}hy\cot hy}$$

$$=\pm\frac{-1}{\operatorname{cosec}hy\sqrt{\operatorname{cosec}h^2y-1}},\quad \operatorname{cosec}h^2y-1=\cot h^2y$$

$$=\pm\frac{-1}{x\sqrt{x^2-1}},\quad \operatorname{cosec}hy=x$$

Here, the sign of the radical is the same as that of cothy which. Here cothy is positive or negative according as x is positive or negative.

$$\frac{d}{dx}(cosech^{-1}x)=\frac{-1}{x\sqrt{x^2+1}}\text{ if } x>0 \text{ and } \frac{d}{dx}(cosech^{-1}x)=\frac{-1}{-x\sqrt{x^2+1}}\text{ if } x<0.$$

Thus $\frac{d}{dx}(cosech^{-1}x)=\frac{-1}{|x|\sqrt{x^2+1}}$ for all values of x.

Derivative of $\coth^{-1}x$: If $y=\cot h^{-1}x$, then $x=\coth y$ The differentiation of $x=\cot hy$ w.r.t y is:

$$\frac{dx}{dy}=-\operatorname{cosec}h^2y$$

Take its reciprocal to obtain the derivative of y w.r.t x:

$$\frac{dy}{dx}=\frac{-1}{cosech^2y}$$
$$=\frac{1}{\coth^2y-1}$$
$$=\frac{-1}{x^2-1},\quad cosech^2y=\coth^2y-1, |x|>1,\ \coth y=x$$

These inverse hyperbolic formulas are listed in the box:

1. $\frac{d}{dx}(\sin h^{-1}x)=\frac{1}{\sqrt{1+x^2}}$
2. $\frac{d}{dx}(\cos h^{-1}x)=\frac{1}{\sqrt{x^2-1}}$
3. $\frac{d}{dx}(\tan h^{-1}x)=\frac{1}{1-x^2}$
4. $\frac{d}{dx}(\operatorname{cosec} h^{-1}x)=\frac{-1}{|x|\sqrt{x^2+1}}$
5. $\frac{d}{dx}(\sec h^{-1}x)=\frac{-1}{x\sqrt{1-x^2}}$
6. $\frac{d}{dx}(\cot h^{-1}x)=\frac{-1}{x^2-1}$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the inverse hyperbolic functions, as summarized in the box:

1. $\frac{d}{dx}(\sin h^{-1}u)=\frac{1}{\sqrt{1+u^2}}\frac{d}{dx}(u)$
2. $\frac{d}{dx}(\cos h^{-1}u)=\frac{1}{\sqrt{u^2-1}}\frac{d}{dx}(u)$
3. $\frac{d}{dx}(\tan h^{-1}u)=\frac{1}{1-u^2}\frac{d}{dx}(u)$
4. $\frac{d}{dx}(\operatorname{cosec} h^{-1}u)=\frac{-1}{|u|\sqrt{u^2+1}}\frac{d}{dx}(u)$
5. $\frac{d}{dx}(\sec h^{-1}u)=\frac{-1}{u\sqrt{1-u^2}}\frac{d}{dx}(u)$
6. $\frac{d}{dx}(\cot h^{-1}u)=\frac{-1}{u^2-1}\frac{d}{dx}(u)$

Example 2.7.2: [Inverse Hyperbolic Function]: Differentiate the following functions:

a. $y=\sinh^{-1}(x^3)$ b. $y=\frac{\sinh^{-1}x}{\cosh^{-1}x}$

Solution:

a. If the given function is $y=\sin h^{-1}(x^3)$, then the derivative of y w.r.t x is:

$$y'=\frac{d}{dx}[\sinh^{-1}(x^3)]=\frac{d}{du}\left(\sin h^{-1}u\right)\frac{d}{dx}(x^3)=\frac{1}{\sqrt{1+u^2}}3x^2=\frac{3x^2}{\sqrt{1+x^6}},\quad u=x^3$$

b. If the given function is $y=\sinh^{-1}x/\cosh^{-1}x$, then the derivative of y w.r.t x is:

$$y'=\frac{d}{dx}\left(\frac{\sin h^{-1}x}{\cosh^{-1}x}\right)$$

$$= \frac{\cosh^{-1}x \frac{d}{dx}(\sinh^{-1}x) - (\sinh^{-1}x)\frac{d}{dx}(\cosh^{-1}x)}{(\cosh^{-1}x)^2}$$

$$\frac{\frac{\cosh^{-1}x}{\sqrt{1+x^2}} - \frac{\sinh^{-1}x}{\sqrt{x^2-1}}}{(\cosh^{-1}x)^2} = \frac{\sqrt{x^2-1}\cosh^{-1}x - \sqrt{1+x^2}\sinh^{-1}x}{\sqrt{1+x^2}\sqrt{x^2-1}(\cosh^{-1}x)^2}$$

[illegible] $\frac{dy}{dx}$ for the following

a. $y = x\ln x^2$ b. $y = \ln(x^2+3x+2)$ c. $y = \ln 5x/x^8$

d. $y = \ln(x^2+1)^{1/2}$ e. $y = 1/[\ln(1+x^2)]$ f. $y = \sqrt[3]{\ln(1-x^2)}$

[illegible] $\frac{dy}{dx}$ for the following

a. [illegible] b. $y = e^{\sqrt{x}}$ c. $y = \left(e^{-x}+e^{x}\right)^2$

d. [illegible] e. $y = xe^{x\ln x}$ f. $y = 5^{(x^2+x)}$

[illegible] $\frac{dy}{dx}$ for the following

a. $y = \log_{10}[3x^2+7]$ b. $y = \log_{10}[x^2+3x+2]$

c. $y = \log_{10}\sqrt{(x^2-7x)+x^3}$ d. $y = \log[\sin(\log x)]$

e. $y = \log_{10}[\sin^{-1}x^2]$ f. $y = \log\tan\left[\frac{1}{2}x+\frac{1}{4}\pi\right]$

[illegible] $\frac{dy}{dx}$ for the following

a. $y = \ln\sqrt{\frac{x+1}{x-1}}$ b. $y = (\cos x)^{\log x}$ c. $y = (1+x^{-1})^x$

d. $y = \frac{(1-x)^{\frac{1}{2}}(2-x^2)^{\frac{2}{3}}}{(3-x^3)^{\frac{3}{4}}(4-x^4)^{\frac{4}{5}}}$ e. $y = \frac{x\sqrt[3]{x^2+4}}{\sqrt{x^2+3}}$ f. $y = (\sin x)(\log x)(x^x)$

5. Use any suitable rule of differentiation to perform $\frac{dy}{dx}$ for the following functions:

a. $y = \cosh(2x^2 + 3x)$

b. $y = e^{\sinh^2 x}$

c. $y = \log(\cos hx)$

d. $y = sech(x^2 + 1) + \tanh(x^2 + 1)$

e. $y = cosech(x^3 + 1)^x$

f. $x \cosh y = y \sin hx + 5$

6. Use any suitable rule of differentiation to perform $\frac{dy}{dx}$ for the following functions:

a. $y = \tanh^{-1}(sinx)$

b. $y = \sinh^{-1}(tanx)$

c. $y = \cosh^{-1}(\sec x)$

d. $y = x \tanh^{-1}(3x)$

e. $y = x\cosh^{-1} x - \sqrt{x^2 - 1}$

f. $\log(\cosh^{-1} x) + \sinh^{-1} y = 6$

7. A research group (used hospital records) developed the approximate mathematical model related to systolic blood pressure and age is:

$p(x) = 40 + 25\ln(x+1), \quad 0 \leq x \leq 65$

where $p(x)$ is the pressure measured in millimeters of mercury and x is age in years. What is the rate of change of pressure at the end of 10 years? at the end of 30 years? at the end of 60 years?

8. A single cholera bacterium divides every 0.5 hour to produce two complete cholera bacteria. If we start with a colony of 5,000 bacteria, then after t hours there will be a $A(t) = 5000\,2^{2t}$ bacteria. Find $A'(t)$, $A'(1)$ and $A'(5)$. Interpret the results.

Glossary

- **Average rate of change:** The average rate of change $y = f(x)$ per unit change in x is given by:

  $$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x)$$
- **The slope of the secant is the average rate of change which measures "the approximate rate of change in phenomena."**
- **Derivative of a Function:** The **instantaneous rate of change** of a function $y=f(x)$ at a particular point $P(x, f(x))$ is the **derivative** of a function $y=f(x)$ at that point,

  $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta) - f(x)}{\Delta x}, \quad y = f(x)$$

  provided this limit exists. This is named **by first principle rule** of derivative of a function f(x).
- **Slope of a Tangent line:** The tangent line to the graph of a function $y=f(x)$ at the point $P(x, f(x))$ is the line through this point having slope

  $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x)$$

  if this limit exists. If this limit does not exist, then there is no tangent (no derivative) at the point.
- **The slope of the tangent is the instantaneous rate of change which measures "the exact rate of change in phenomena."**
- In business terminology,
  - the instantaneous rate of change of cost (or profit or revenue) is the **marginal cost** which counts "as the approximate rate of change in business phenomena".
  - the average rate of change is the exact rate of change which counts "as the actual rate of change in business phenomena"
- **Power Rule:** For any real number n, if $f(x) = x^n$, then:

  $$f'(x) = nx^{n-1}$$
- **Chain Rule**

  If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f[g(x)]$ is a differentiable function of x and

  $$\frac{dy}{dx} = \frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)]\frac{d}{dx}[g(x)] = \frac{dy}{du}\frac{du}{dx}, \quad \textit{Leibniz notation}$$
- **Differentiation of Parametric Functions:** Consider two differential functions $x = f(t)$ and $y = g(t)$ of parameter t. If $t = h(x)$ is an inverse function of $x = f(t)$, then $y = g[h(x)]$ is a function of x.
- **Generalized Power Rule**

  Let $f(x)$ be a differentiable function of x. If $y = [f(x)]^n$, then:

  $$\frac{dy}{dx} = n[f(x)]^{n-1} \cdot \frac{d}{dx}[f(x)], \quad \text{where } n \text{ is any real number.}$$

UNIT 3

HIGHER ORDER DERIVATIVES AND APPLICATIONS

This unit tells us, how to:

- find higher order derivatives of a function such as algebraic, trigonometric, exponential and logarithmic functions.
- find the second order derivative of implicit function, inverse trigonometric and parametric functions.
- define Taylor's and Maclaurin's theorems and the use of these theorems to expand sinx, cosx, tanx, $a^x, e^x, \log_a(1+x)$ and $\ln(1+x)$.
- consider applications of derivative and its geometric view to develop the tangent, and normal equations and the angle of intersection in between the two curves.
- introduce maxima and minima and the topics related to maxima and minima, such as increasing and decreasing functions, extreme values of a function, first and second derivative rules and the use of second derivative rule to examine the extreme values of a function.
- solve real life problems related to extreme values.

3.1 Higher Order Derivatives

If a function $y = f(x)$ has a first derivative y', then the derivative of y', if it exists, is the **second derivative** of $y = f(x)$, written y''. The derivative of y'', if it exists, is called the **third derivative** of $y = f(x)$, written as y'''. By continuing this process, we can find **third derivative** and other **higher derivatives**. For example, if $f(x) = x^4+2x^3+3x^2-5x+7$, then the higher derivatives are the following:

$$\frac{dy}{dx} = 4x^3+6x^2+6x-5, \qquad \textit{first derivative of } y$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= 12x^2+12x+6, \qquad \textit{second derivative of } y$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= 24x+12, \qquad \textit{third derivative of } y$$

The notations for indicating higher order derivatives are displayed in the box:

The second derivative of $y = f(x)$ can be written with any of the following notations: $\frac{d^2y}{dx^2}$, y'', $f''(x)$, $D^2_x[f(x)]$

The third derivative can be written in a similar way. For derivative $n \geq 4$, the derivative holds the notation $f^{(n)}(x)$, $n = 4, 5, ...$

Example 3.1.1:[Second Derivative]: Find the second derivative of the following functions:

a. $y = 8x^3 - 9x^2 + 6x + 4$ b. $y = \frac{4x+2}{3x-1}$

Solution:

a. If the given function is $y = 8x^3 - 9x^2 + 6x + 4$ then, the first and second derivatives of the given function through linearity property are the following:

$$\frac{dy}{dx} = \frac{d}{dx}(8x^3 - 9x^2 + 6x + 4) = 24x^2 - 18x + 6$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(24x^2 - 18x + 6) = 48x - 18$$

b. If the given function is $y = \frac{u}{v} = \frac{4x+2}{3x-1}$, then the first and second derivatives of the given function through quotient rule are the following:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - \frac{dv}{dx}u}{v^2} = \frac{(4)(3x-1)-(3)(4x+2)}{(3x-1)^2} = \frac{-10}{(3x-1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{\frac{du}{dx}v - \frac{dv}{dx}u}{v^2}$$

$$= \frac{(0)(3x-1)^2 - (-10)(2)(3x-1)(3)}{(3x-1)^4}$$

$$= \frac{60(3x-1)}{(3x-1)^4} = \frac{60}{(3x-1)^3}$$

In the previous unit, we saw that the first derivative o a n the rate of change of the function. The second derivative, then, represents change of the first derivative. If a function describes the position of a (along a straight line) at time t, then the first derivative giv s the object. That is, if y=s(t) describes the position (along a straight line) of time t, then $v(t)=s'(t)$ gives the velocity at a time t.

The rate of change of velocity is called acce e a o Sin th derivative gives the rate of change of the first derivative, the a l t derivative of the velocity Thus, if a(t) represents the acc le ation at

$$a(t)=\frac{dv}{dt}=\frac{d^2s}{dt^2}=s''(t)$$

Example 3.1.2: An ob ect is mo ng al ng e w i feet) at time t (in seconds):

$s(t)=t^3-2t^2-7t+9$

a. Find the velocity at any time t. b. in the acc leration a

c. The object stops when velocity is zero. or $t\geq 0$,

Solution:

a. The velocity at any time t is the first derivative f s(t w

$$v=\frac{ds}{dt}=3t^2-4t-7$$

b. The acceleration at any time t is the first derivative f v(

$$a=\frac{dv}{dt}=6t-4$$

c. Use v(t)=0 to obtain the time:

$$3t^2-4t-7=0$$

$$(3t-7)(t+1)=0,\ t=-1,\ 7/3$$

The object will stop at 7/3 seconds, since we want time $t>0$

i) *Higher order derivatives of algebraic, and trigonometric functions*

The successive derivatives of some functions are gathered to obtain the general form of nth derivatives in the following cases:

The nth derivative of $f(x)=(ax+b)^m$: If $f(x)=(ax+b)^m$, m is a positive integer then the successive derivatives of the given function developed a general term for the nth derivative of a function:

$$\begin{aligned} f(x) &= (ax+b)^m \\ f'(x) &= ma(ax+b)^{m-1} \\ f''(x) &= m(m-1)a^2(ax+b)^{m-2} \\ &\vdots \\ f^n(x) &= m(m-1)(m-2)\ldots(m-n+1)(a^n)(ax+b)^{m-n} \\ &= \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}, \text{ if } m \text{ is positive integer} \end{aligned} \qquad (1)$$

If $m=-1$, then the nth derivative of $f(x)=\frac{1}{(ax+b)}$ is obtained by inserting $m=-1$ in equation (1):

$$\begin{aligned} f^n(x) &= (-1)(-2)(-3)\ldots(-n)(a^n)(ax+b)^{-1-n} \\ &= \frac{(-1)^n\, n!a^n}{(ax+b)^{n+1}} \end{aligned} \qquad (2)$$

The nth derivative of $f(x)=\ln(ax+b)$: If $f(x)=\ln(ax+b)$, then the successive derivatives developed a general term for the nth derivative of a function:

$$\begin{aligned} f(x) &= \ln(ax+b) \\ f'(x) &= \frac{a}{ax+b} \\ f''(x) &= \frac{(-1)a^2}{(ax+b)^2} \\ f'''(x) &= \frac{(-1)^2 1.2.a^3}{(ax+b)^3} \\ &\vdots \\ f^n(x) &= (-1)(-2(-3)(-4)\ldots(-(n-1))(a^n)(ax+b)^{-n} \\ &= \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n} \end{aligned} \qquad (3)$$

The nth derivative of $f(x)=a^{mx}$**:** If $f(x)=a^{mx}$, then the successive derivatives developed a general term for the nth derivative:

$$
\begin{aligned}
f(x)&=a^{mx}\\
f'(x)&=a^{mx}\log a\frac{d}{dx}(mx)=ma^{mx}\log a\\
f''(x)&=m\log a\frac{d}{dx}(a^{mx})=m\log a(a^{mx})\log a\frac{d}{dx}(mx)\\
&=m^2a^{mx}(\log a)^2\\
&\vdots\\
f^{n}(x)&=m^na^{mx}(\log a)^n
\end{aligned}
\tag{4}
$$

If $a=e$, then the nth derivative of $f(x)=e^{mx}$ is obtained by inserting $a=e$ in equation (4):

$$
\begin{aligned}
f(x)&=e^{mx}\\
f'(x)&=me^{mx}\\
f''(x)&=m(m)e^{mx}=m^2e^{mx}\\
&\vdots\\
f^{n}(x)&=m^ne^{mx}
\end{aligned}
\tag{5}
$$

The nth derivative of $f(x)=\sin(ax+b)$**:** If $f(x)=\sin(ax+b)$, then the successive derivatives developed a general term for the nth derivative of a function:

$$
\begin{aligned}
f(x)&=\sin(ax+b)\\
f'(x)&=a\cos(ax+b)=a\sin(ax+b+\frac{\pi}{2})\\
f''(x)&=a^2\cos(ax+b+\frac{\pi}{2})=a^2\sin(ax+b+\frac{2\pi}{2})\\
f'''(x)&=a^3\cos(ax+b+\frac{2\pi}{2})=a^3\sin(ax+b+\frac{3\pi}{2})\\
&\vdots\\
f^{n}(x)&=a^n\sin\left[ax+b+\frac{n\pi}{2}\right]
\end{aligned}
\tag{6}
$$

The nth derivative of $f(x)=\cos(ax+b)$: If $f(x)=\cos(ax+b)$, then the successive derivatives developed a general term for the n–th derivative:

$$f(x)=\cos(ax+b)$$

$$f'(x)=-a\sin(ax+b)=a\cos(ax+b+\frac{\pi}{2})$$

$$f''(x)=-a^2\sin(ax+b+\frac{\pi}{2})=a^2\cos(ax+b+\frac{2\pi}{2})$$

$$f'''(x)=-a^3\sin(ax+b+\frac{2\pi}{2})=a^3\cos(ax+b+\frac{3\pi}{2})$$

$$\vdots$$

$$f^n(x)=a^n\cos\left[ax+b+\frac{n\pi}{2}\right] \qquad (7)$$

Example 3.1.3: [5th Derivative]: Find the 5th derivatives of the following functions:

a. $f(x)=(6x+4)^9$ b. $f(x)=1/4x+3$ c. $f(x)=\ln(4x+7)$

d. $f(x)=6^{4x}$ e. $f(x)=e^{4x}$ f. $f(x)=\sin(5x+7)$

Solution:

a. If $f(x)=(6x+4)^9$ with $a=6$, $b=4$ and $m=9$, then the 5–th derivative of the given function is obtained by inserting n=5 in equation (1):

$$f^5(x)=\frac{9!}{(9-5)!}6^5(6x+4)^{9-5}$$

$$=\frac{6^5 9!}{4!}(6x+4)^4$$

$$=6^5(9)(8)(7)(6)(5)(6x+4)^4=6^5\left(15120(6x+4)^4\right)$$

b. If $f(x)=\frac{1}{(4x+3)}$ with $a=4$ and $b=3$, then the 5th derivative of the given function is obtained by inserting $n=5$ in equation (2):

$$f^5(x)=\frac{(-1)^5 5!4^5}{(4x+3)^6}=\frac{-4^5 5!}{(4x+3)^6}$$

c. If $f(x)=\ln(4x+7)$ with $a=4$ and $b=7$, then the 5–th derivative of the given function is obtained by inserting n=5 in equation (3):

$$f^5(x)=\frac{(-1)^4 4!4^5}{(4x+7)^5}=\frac{4^5 4!}{(4x+7)^5}$$

d. If $f(x)=6^{4x}$ with $a=6$ and $m=4$, then the 5th derivative of the given function is obtained by inserting n=5 in equation (4):

$$f^5(x)=4^5 6^{4x}(\log 6)^5$$

e. If $f(x)=e^{4x}$ with $m=4$, then the 5th derivative of the given function is obtained by inserting $n=5$ in equation (5):

$$f^5(x)=4^5 e^{4x}$$

f. If $f(x)=\sin(5x+7)$ with $a=5$ and $b=7$, then the 5th derivative of the given function is obtained by inserting $n=5$ in equation (6):

$$f^5(x)=5^5\sin\left[5x+7+\frac{5\pi}{2}\right]$$

Example 3.1.4:[Algebraic Function]: Find the first three derivatives of the following functions:

(a) $y=x^7+6x^5+9x^4+4x^3+2x+1$ (b) $p(t)=(t^2+2)\sin t$

(c). $f(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}$

Solution:

(a) If the given polynomial is $y=x^7+6x^5+9x^4+4x^3+2x+1$, then the first three derivatives of the polynomials are the following:

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^7+6x^5+9x^4+4x^3+2x+1\right)=7x^6+30x^4+36x^3+12x^2+2$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(7x^6+30x^4+36x^3+12x^2+2\right)=42x^5+120x^3+108x^2+24x$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(42x^5+120x^3+108x^2+24x\right)=210x^4+360x^2+216x+24$$

b. If the given function is, $p(t)=uv=(t^2+2)\sin t$, then the first two derivatives of the trigonometric functions are the following:

$$\frac{dp}{dt}=u\frac{dv}{dt}+v\frac{du}{dt}$$

$$=(t^2+t)\frac{d}{dt}(\sin t)+(\sin t)\frac{d}{dt}(t^2+t)=(t^2+t)\cos t+(2t+1)\sin t$$

$$\frac{d^2p}{dt^2}=\frac{d}{dt}[(t^2+t)cost+(2t+1)\sin t]$$

$$=\frac{d}{dt}[(t^2+t)\cos t]+\frac{d}{dt}[(2t+1)\sin t]$$

$$=-(t^2+t)sint+(2t+1)cost+(2t+1)cost+2\sin t$$

$$=-(t^2+t-2)\sin t+(4t+2)\cos t$$

c. If the given function is $f(x)=\frac{u}{v}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$, then, the first two derivatives of the given function are the following:

$$\frac{df}{dx}=\frac{d}{dx}(\frac{u}{v})=\frac{\frac{du}{dx}v-\frac{dv}{dx}u}{v^2}$$

$$=\frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$

$$=\frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$$

$$=\frac{e^{2x}+e^{-2x}+2-e^{2x}-e^{-2x}+2}{(e^x+e^{-x})^2}=\frac{4}{(e^x+e^{-x})^2}$$

$$\frac{d^2f}{dx^2}=\frac{(0)(e^x+e^{-x})^2-2(e^x+e^{-x})(e^x-e^{-x})(4)}{(e^x+e^{-x})^4}$$

$$=\frac{-8(e^x+e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^4}$$

$$=\frac{-8(e^x-e^{-x})}{(e^x+e^{-x})^3}$$

ii) *Second derivatives of implicit, inverse trigonometric and parametric functions*

Example 3.1.5: [Implicit Function]: Find the second derivative of $x^2y+2y^3=3x+2y$.

Solution:

a. The equation is

$$x^2y+2y^3=3x+2y \tag{8}$$

The first implicit derivative of (8) w.r.t x is:

$$\frac{d}{dx}\left(x^2y+2y^3\right)=\frac{d}{dx}(3x+2y)$$

$$\frac{d}{dx}(x^2y)+\frac{d}{dx}(2y^3)=\frac{d}{dx}(3x)+\frac{d}{dx}(2y)$$

$$2xy+x^2\frac{dy}{dx}+6y^2\frac{dy}{dx}=3+2\frac{dy}{dx} \tag{9}$$

$$(x^2+6y^2-2)\frac{dy}{dx}=3-2xy$$

The second implicit derivative of first implicit derivative (9) w.r.t x is :

$$\frac{d}{dx}\left[(x^2+6y^2-2)\frac{dy}{dx}\right]=\frac{d}{dx}(3-2xy)$$

$$\frac{d}{dx}[(x^2+6y^2-2)]\frac{dy}{dx}+(x^2+6y^2-2)\frac{d}{dx}(\frac{dy}{dx})=\frac{d}{dx}(3)-\frac{d}{dx}(2xy)$$

$$(2x+12y\frac{dy}{dx}-0)\frac{dy}{dx}+(x^2+6y^2-2)\frac{d^2y}{dx^2}=0-2y-2x\frac{dy}{dx}$$

$$(2x+12y\frac{dy}{dx})\frac{dy}{dx}+(x^2+6y^2-2)\frac{d^2y}{dx^2}=-2y-2x\frac{dy}{dx}$$

$$\left(x^2+6y^2-2\right)\frac{d^2y}{dx^2}=-2y-(2x+2x)\frac{dy}{dx}-12y\left(\frac{dy}{dx}\right)^2$$

$$\frac{d^2y}{dx^2}=\frac{-2\left(y+2x\frac{dy}{dx}+6y\left(\frac{dy}{dx}\right)^2\right)}{\left(x^2+6y^2-2\right)}$$

Example 3 1 6: [Inverse Trigonometric Function]: Find the second derivative of $\cos^{-1}y+y=2xy$.

Solution:

a. The given equation is

$\cos^{-1}y+y=2xy$ (10)

The first implicit derivative of (12) w.r.t x is:

$$\frac{d}{dx}\left(\cos^{-1}y+y\right)=\frac{d}{dx}(2xy)$$

$$\frac{d}{dx}(\cos^{-1}y)+\frac{d}{dx}(y)=2\frac{d}{dx}(xy)$$

$$\frac{-1}{\sqrt{1-y^2}}\frac{dy}{dx}+\frac{dy}{dx}=2\left(\frac{d}{dx}(x)y+x\frac{d}{dx}(y)\right) \quad (11)$$

$$\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{dy}{dx}=2y+2x\frac{dy}{dx}$$

The second implicit derivative of first implicit derivative (11) w.r.t x is :

$$\frac{d}{dx}\left[\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{dy}{dx}\right]=\frac{d}{dx}\left(2y+2x\frac{dy}{dx}\right)$$

$$\frac{d}{dx}\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{dy}{dx}+\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{d^2y}{dx^2}=2\frac{dy}{dx}+2\frac{dy}{dx}+2x\frac{d^2y}{dx^2}$$

$$\left(\frac{-2y}{2(1-y^2)^{3/2}}+0\right)\frac{dy}{dx}+\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{d^2y}{dx^2}=4\frac{dy}{dx}+2x\frac{d^2y}{dx^2}$$

$$\left(\frac{-1}{\sqrt{1-y^2}}+1-2x\right)\frac{d^2y}{dx^2}=\left(\frac{y}{(1-y^2)^{3/2}}+4\right)\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{\left(\frac{y}{(1-y^2)^{3/2}}+4\right)\frac{dy}{dx}}{\left(\frac{-1}{\sqrt{1-y^2}}+1-2x\right)}$$

Example 3.1.7: [Parametric Functions]: Find the second derivative $\frac{d^2y}{dx^2}$, when the parametric functions are $x(t)=1+t^2, y(t)=t^3+2t^2+1$.

Solution: The first derivative of the parametric functions x=x(t) and y=y(t) w.r.t x is:

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad (12)$$

The second derivative of the parametric functions is obtained by taking the derivative of equation (12):

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$=\frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dt}, \quad \textit{Multiply and divide out by dt}$$

$$=\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx} \qquad (13)$$

The quotient rule of differentiation is used to simplify the right hand side of equation (13):

$$\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)=\frac{\frac{dx}{dt}\frac{d}{dt}\left(\frac{dy}{dt}\right)-\frac{dy}{dt}\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}$$

$$=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2} \qquad (14)$$

Use (14) in (13) to obtain the general term for second derivative of parametric functions $x(t)$ and $y(t)$:

$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}\frac{dt}{dx}$$

$$=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3},\ \textit{replace}\ \ dt/dx=\frac{1}{\frac{dx}{dt}} \qquad (15)$$

In light of result (15), the first and second derivatives of the parametric functions $x(t)=1+t^2$, $y(t)=t^3+2t^2+1$ with

$$\frac{dx}{dt}=2t,\ \frac{d^2x}{dt^2}=2,\ \frac{dy}{dt}=3t^2+4t,\ \frac{d^2y}{dt^2}=6t+4$$

are the following:

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}$$

$$=\frac{3t^2+4t}{2t}$$

$$=\frac{t(3t+4)}{2t}$$

$$=\frac{3t+4}{2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

$$= \frac{(2t)(6t+4)-(3t^2+4t)(2)}{8t^3}$$

$$= \frac{12t^2+8t-6t^2-8t}{8t^3}$$

$$= \frac{6t^2}{8t^3} = \frac{3}{4t}$$

Result (15) is obtained through right click on the last end of the expression by selecting "Differentiate < x" on the context menu. For second derivative, click on the last end of the expression of result (15) by selecting again "Differentiate < x"to obtain the required second derivative. For derivatives higher than two, repeat the process again and again to obtain the required higher order derivatives.

Exercise 3.1

1. Find the indicated higher derivatives of the following functions:

a. $f(x) = 3x^3 + 4x + 5, \ f'''(x)$

b. $f(x) = x + \frac{1}{x}, \ f'''(x)$

c. $f(x) = 1 + \frac{2}{x} - \frac{3}{x^2}, \ f''(x)$

d. $s(t) = \sqrt{5t+7}, \ s''(t)$

e. $y = \frac{x+1}{x-1}, \ y''$

f. $y = (x+3)(x^2+7x+2), \ y''$

2. Find the indicated higher derivatives of the following trigonometric functions:

a. $y = \tan x, \ y''''$

b. $y = \ln \sin x, \ y'''$

c. $y = \sqrt{\sec 2x}, \ y''''$

d. $y = \frac{1}{x}, \ y^{ix}$

e. $y = \sin(\sin x), \ y''$

3. Use implicit rule to find out the second derivative of the following functions:

a. $y = x + \arctan y$

b. $x^2 + y^2 = r^2,$

c. $y^2 - 2xy = 0, \ y''$

d. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$

c. $\sec x \cos y = C$,   f. $e^x + x = e^y + y$,

4. Use parametric differentiation to find out d^2y/dx^2 for the following parametric functions $x(t)$ and $y(t)$:

a. $x = 4t^2 + 1, \ y = 6t^3 + 1$ b. $x = 3at^2 + 2, \ y = 6t^4 + 9$

c. $x = a(t - \sin t), \ y = a(1 - \cos t)$ d. $x = a\cos 2t, \ y = b\sin 2t$

e. $x = \dfrac{3at}{1+t^3}, \ y = \dfrac{3at^2}{1+t^3}$ f. $x = a\dfrac{1-t^2}{1+t^2}, \ y = b\dfrac{2t}{1+t^2}$

3.2 Taylor's and Maclaurin's Expansions

Often the value of a function and the values of its derivatives are known at a particular point and from this information it is desired to obtain values of the function around that particular point. The Taylor polynomials and Taylor series allow us to make such estimates.

i) *Maclaurin's and Taylor's theorems. Use of these theorems to expand sinx, cosx, tanx,* $a^x, e^x, \log_a(1+x), \ln(1+x)$

If $f(x)$ and its n derivatives at $x = x_0$ are $f'(x_0), f''(x_0), \ldots, f^n(x_0)$, then the n-th order Taylor polynomial $p_n(x)$ may be written as:

$$p_n(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \ldots\ldots + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) \quad (16)$$

This polynomial provides an approximation to $f(x)$. The polynomial and its n derivatives are very much matched with the values of $f(x)$ and its first n derivatives evaluated at $x = x_0$:

$$p_n(x_0) = f(x_0), \ p_n'(x_0) = f'(x_0), \ p_n''(x_0) = f''(x_0), \ldots, \ p_n^n(x_0) = f_n^{(n)}(x_0)$$

Example 3.2.1: The function $y = f(x) = e^x$ and its derivatives evaluated at $x_0 = 0$ are known by $f(0) = 1, \ f'(0) = 1, \ f''(0) = 1, \ f'''(0) = 1, \ f^4(0) = 1$. Use fourth order Taylor polynomial about $x_0 = 0$ to estimate f (0.2) at $x = 0.2$

Solution:

The fourth order Taylor polynomial $p_4(x)$ is obtained by terminating the Taylor polynomial (16) after fourth order derivative term:

$$p_4(x)= f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\frac{(x-x_0)^3}{3!}f'''(x_0)+\frac{(x-x_0)^4}{4!}f''''(x_0) \quad (17)$$

Insert $x_0=0$ in (17) to obtain:

$$p_4(x)= f(0)+f'(0)x+f''(0)\frac{x^2}{2!}+f'''(0)\frac{x^3}{3!}+f^4(0)\frac{x^4}{4!}$$

$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}, \quad (18)$$

The Taylor polynomial (18) is used to obtain approximation of a function $y=f(x)=e^x$ at x=0.2:

$$p_4(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$$

$$p_4(0.2)=1+0.2+\frac{(0.2)^2}{2!}+\frac{(0.2)^3}{3!}+\frac{(0.2)^4}{4!}$$

$$=1+0.2+0.02+0.00133+0.00007=1.2214$$

Notice that the he Taylor polynomial approximation equals the actual function value $y=f(0.2)=e^{0.2}=1.2214$ at x=0.2.

Taylor's Series: The Taylor polynomials have been used to estimate the values of $y=f(x)$ at various x values. It is reasonable to ask:

1. How accurate Taylor polynomials generated by $y=f(x)$ at x_0 to approximate $y=f(x)$ at values of x other than x_0?
2. If more and more terms are used in the Taylor polynomial, then this will produce a better and better approximation to $y=f(x)$.

To answer these questions, we introduce the Taylor series. As more and more terms are included in the Taylor polynomial, we obtain an infinite series, known as a

Taylor series:

$$p(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+......+\frac{(x-x_0)^n}{n!}f^n(x_0) \quad (19)$$

For some Taylor series, the value of the series equals the value of the function for every value of x. That is, the Taylor series approximations of e^x, $\sin x$ and $\cos x$ equal the values of e^x, $\sin x$ and $\cos x$ for every value of x. However, some functions have a Taylor series which equals the function only for a limited range of x values. For example, the value of a function $f(x)=1/(1+x)$ which equals its Taylor series only when $-1<x<1$.

Maclaurin's Series: A special case of a Taylor series occurs, when the function $y=f(x)$ is known only at the origin $x_0=0$.. This special condition imposed on Taylor series, develops the **Maclaurin's series:**

$$p(x)=f(0)+xf'(0)+\frac{(x)^2}{2!}f''(0)+......+\frac{(x)^n}{n!}f^n(0) \quad (20)$$

The Taylor and Maclaurin's series of $y=f(x)$ about a particular point x_0 are of course:

$$f(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)$$
$$+\frac{(x-x_0)^3}{3!}f'''(x_0)+...+\frac{(x-x_0)^n}{n!}f^n(x_0) \quad (21)$$

$$f(x)=f(0)+xf'(0)+\frac{(x)^2}{2!}f''(0)+\frac{(x)^3}{3!}f'''(0)......+\frac{(x)^n}{n!}f^{(n)}(0) \quad (22)$$

If we use $x-x_0=h$, then equations (21) and (22) take the popular notation for the Taylor and Maclaurin's series of order n:

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+....+\frac{h^n}{n!}f^{(n)}(x_0) \quad (23)$$

$$f(x_0+h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(0)+\frac{h^3}{3!}f'''(0)+....+\frac{h^n}{n!}f^{(n)}(0) \quad (24)$$

The graphical view of a function y=f(x) at $x=x_0$ is shown in the fig .3.1:

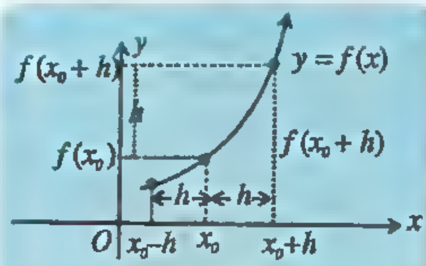


Fig. 3.1

The popular notation for the Taylor & Maclaurin's series of order n are:

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+.....+\frac{h^n}{n!}f^{n}(x_0)$$

$$f(x_0+h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(0)+\frac{h^3}{3!}f'''(0)+........+\frac{h^n}{n!}f^{n}(0)$$

If a function $y=f(x)$ is known at a particular point $x_0 \neq 0$, then the Taylor series (23) at a forward or backward point $x=x_0 \pm h$ of a function y=f(x) are:

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+....,\ x=x_0+h$$

$$f(x_0+h)=f(x_0)-hf'(x_0)+\frac{h^2}{2!}f''(x_0)-\frac{h^3}{3!}f'''(x_0)+....,\ x=x_0-h$$

Example 3.2.2: [Taylor Series of e^x]: Use Taylor's series to approximate the value of a function $f(x)=e^x$ at a point $x_0=2$.

Solution: The function and its derivatives at $x_0=2$

$$f(x)=e^x,\ f(2)=e^2=7.3891,\quad f'(x)=e^x,\ f'(2)=e^2=7.3891$$
$$f''(x)=e^x,\ f'''(2)=e^2=7.3891$$

are used in Taylor series (21) to obtain the Taylor series approximation of e^x at a point $x_0=2$:

$$e^x=f(2)+(x-2)f'(2)+\frac{(x-2)^2}{2!}f''(2)+\frac{(x-2)^3}{3!}f'''(2)+.....$$

$$=7.3891+7.3891(x-2)+7.3891\frac{(x-2)^2}{2!}+7.3891\frac{(x-2)^3}{3!}+.....$$

Example 3.2.3: [Maclaurin Series of e^x]: Use Maclaurin's series to approximate the value of a function $f(x)=e^x$ at a point $x_0=0$

Solution: The function and its derivatives at $x_0=0$

$$f(x)=e^x,\ f(0)=1,\quad f'(x)=e^x,\ f'(0)=1,\quad f''(x)=e^x,\ f'''(o)=1$$

are used in Maclaurin series (22) to obtain the Maclaurin's series approximation of e^x at a point $x_0=0$:

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots$$

Example 3.2.4: [Maclaurin Series of a^x]: Use Maclaurin's series to approximate the value of a function $f(x)=a^x$ at a point $x_0=0$.

Solution: The function and its derivatives at $x_0=0$

$$f(x)=a^x,\ f(0)=1,\quad f'(x)=a^x\log_e a,\ f'(0)=\log_e a$$

$$f''(x)=a^x(\log_e a)^2,\ f''(0)=(\log_e a)^2$$

are used in Maclaurin series (22) to obtain the Maclaurin series approximation of a^x at a point $x_0=0$:

$$a^x=f(0)+x\,f'(0)+\frac{(x)^2}{2!}f''0)+\frac{(x)^3}{3!}f'''(0)+\ldots$$

$$=1+x\log_e a+\frac{x^2}{2!}(\log_e a)^2+\frac{x^3}{3!}(\log_e a)^3+\ldots..$$

Example 3.2.5: [Maclaurin Series of sinx]: Use Maclaurin's series to approximate the value of a function f(x)=sinx at a point $x_0=0$.

Solution: The function and its derivatives at $x_0=0$

$$f(x)=\sin x,\ f(0)=\sin(0)=0,\quad f'(x)=\cos x,\ f'(0)=\cos(0)=1$$

$$f''(x)=-\sin x,\ f''(0)=-\sin(0)=0,\quad f'''(x)=-\cos x,\ f'''(0)=-\cos(0)=-1$$

are used in Maclaurin series (22) to obtain the Maclaurin series approximation of $\sin x$ at a point $x_0=0$:

$$\sin x=f(0)+(x)\,f'(0)+\frac{(x)^2}{2!}f''(0)+\frac{(x)^3}{3!}f'''(0)+\ldots..$$

$$=0+x-0-\frac{(x)^3}{3!}+\ldots\ldots=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots.$$

Example 3.2.6: [Maclaurin's Series of $\cos x$]: Use Maclaurin's series to approximate the value of a function $f(x)=\cos x$ at a point $x_0=0$.

Solution: The function and its derivatives at $x_0=0$.

$$f(x)=\cos x,\ f(0)=\cos 0=1,\quad f'(x)=-\sin x,\ f'(0)=-\sin 0=0$$
$$f''(x)=-\cos x,\ f''(0)=-\cos 0=-1,\quad f'''(x)=\sin x,\ f'''(0)=\sin 0=0$$

are used in Maclaurin series (22) to obtain the Maclaurin series approximation of a function $\cos x$ at a point $x_0=0$:

$$\cos x=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots..$$
$$=1+x(0)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(0)+\ldots...$$
$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots..$$

Example 3.2.7: [Maclaurin's Series of tanx]: Use Maclaurin's series to approximate the value of a function $f(x)=\tan x$ at a point $x_0=0$

Solution: The function and its derivatives at $x_0=0$

$$f(x)=\tan x,\ f(0)=\tan 0=0,\ f'(x)=\sec^2 x,\ f'(0)=\sec^2 0=1$$
$$f''(x)=2\sec^2 x\tan x,\ f''(0)=2(1)(0)=0,\ f'''(x)=2\sec^4 x+4\tan^2 x\sec^2 x$$
$$f'''(0)=2\sec^4 0+4\tan^2 0\sec^2 0=2$$

are used in Maclaurin's (22) to obtain the Maclaurin's series approximation of a function $\tan x$ at a point $x_0=0$:

$$\tan x=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots..$$
$$=0+x(1)+\frac{x^2}{2!}(0)+\frac{x^3}{3!}(2)\ldots\ldots=x+2\frac{x^3}{3!}+\ldots\ldots$$

Example 3.2.8: [Maclaurin's Series of ln(1+x)]: Use Maclaurin's series to approximate the value of a function $f(x)=\ln(1+x)$ at a point $x_0=0$.

Solution: The function and its derivatives at $x_0=0$

$$f(x)=\ln(1+x),\ f(0)=\ln(1)=0,\quad f'(x)=1/(1+x),\ f'(0)=1$$
$$f''(x)=-1/(1+x)^2,\ f''(0)=-1,\quad f'''(x)=2/(1+x)^3,\ f'''(0)=2$$

are used in Maclaurin's series (22) to obtain the Maclaurin's series approximation of $\ln(1+x)$ at a point $x_0 = 0$:

$$\ln(1+x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$$= 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \ldots\ldots$$

$$= x - \frac{x^2}{2!} + 2\frac{x^3}{3!} + \ldots\ldots$$

Example 3.2.9: [Maclaurin's Series of $\log_a(1+x)$]: Use Maclaurin's series to approximate the value of a function $f(x) = \log_a(1+x)$ at a point $x_0 = 0$.

Solution: The function and its derivatives at $x_0 = 0$

$$f(x) = \log_a(1+x),\ f(0) = \log_a(1),$$

$$f'(x) = \frac{1}{1+x}\log_a e,\ f'(0) = \log_a e = \log_a e$$

$$f''(x) = -\frac{1}{(1+x)^2}\log_a e,\ f''(0) = -\log_a e$$

$$f'''(x) = \frac{2}{(1+x)^3}\log_a e, \qquad f'''(0) = 2\log_a e$$

are used in Maclaurin's series (22) to obtain the Maclaurin's series approximation of a function $f(x) = \log_a(1+x)$ at a point $x_0 = 0$:

$$\log_a(1+x) = f(0) + (x) f'(0) + \frac{(x)^2}{2!} f''(0) + \frac{(x)^3}{3!} f'''(0) + \ldots\ldots$$

$$= \log_a(1) + x\log_a e - \frac{(x)^2}{2!}\log_a e + 2\frac{x^3}{3!}\log_a e + \ldots\ldots$$

3.3 Application of Derivatives

In this section, we shall see how to use derivatives to determine the tangent, and normal lines, the angles in between two curves, the maximum and minimum values of a function as well as the intervals where the function is increasing or decreasing.

i) Geometrical interpretation of derivative

Consider a function $y = f(x)$ as shown in the Fig. 3.2 below:

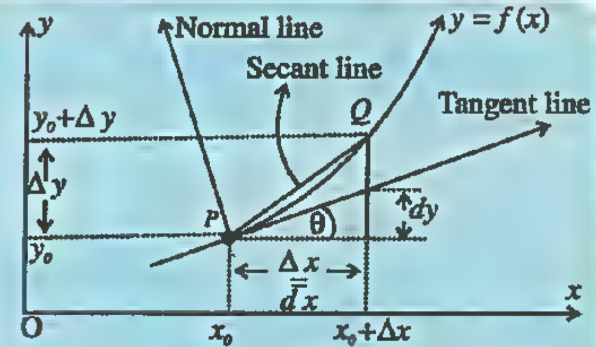


Fig. 3.2

Let $P(x_0, y_0)$ be a point on a curve $y = f(x)$. The change Δx in x develops a change Δy in y. The coordinates of a point Q are therefore $Q(x_0 + \Delta x, y_0 + \Delta y)$. Notice that the slope of the secant line PQ is:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \qquad (25)$$

If we take values of Q closer to P, then Q approaches P, and Δx approaches 0 and the slope of the secant line PQ automatically approaches the slope of the tangent line at a particular point P and is denoted by:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0) \qquad (26)$$

ii) Equations of tangent and normal lines at a given point

If the slope of the tangent line on a curve $y = f(x)$ at a particular point $P(x_0, y_0)$ is $f'(x_0)$, then the tangent line on this curve at a particular point $P(x_0, y_0)$ is the nonhomogeneous line (developed from the definition of the point form of the straight line):

$$\begin{aligned} y - y_0 &= f'(x_0)(x - x_0) \\ y - y_0 &= m(x - x_0), \qquad m = f'(x_0) \end{aligned} \qquad (27)$$

The normal line is the line perpendicular to the tangent line on this curve at a particular point $P(x_0, y_0)$ with slope $\frac{-1}{f'(x_0)}$:

$$(y - y_0) = \frac{-1}{f'(x_0)}(x - x_0)$$

$$(y - y_0) = -\frac{1}{m}(x - x_0), \qquad m = f'(x_0) \tag{28}$$

Example 3.3.1: [Tangent and Normal Equations]: Find the equations of the tangent and normal lines on a curve $y = x^2$ at a point $P(2, 4)$.

Solution: If the given curve is $y = x^2$, then, the slope of the tangent line is the first derivative of the given curve at a particular point $P(2, 4)$:

$$f'(x) = 2x$$

$$f(2) = 2(2) = 4 = m, \text{ say, at the point } P(2,4)$$

The equation of a tangent line (27) on the given curve at a particular point P(2, 4) is:

$$y - y_0 = m(x - x_0)$$

$$(y - 4) = 4(x - 2)$$

$$-4x + y - 4 + 8 = 0 \Rightarrow -4x + y + 4 = 0 \Rightarrow 4x - y - 4 = 0$$

The equation of a normal line (28) on the given curve at a particular point P(2, 4) is:

$$(y - y_0) = \frac{-1}{m}(x - x_0)$$

$$(y - 4) = \frac{-1}{4}(x - 2)$$

$$4(y - 4) = -(x - 2)$$

$$x + 4y - 16 - 2 = 0$$

$$x + 4y - 18 = 0$$

Example 3.3.2: Tangent and Normal Equations]: Find the equations of the tangent and normal lines on the curve $y = 9 - x^2$ at a point, when y crosses the x– axis.

Solution: The coordinates of a particular point P at which the given curve $y = 9 - x^2$ crosses the x– axis are

$$y = 0$$

Put y=0 in $y = 9 - x^2$ to obtain a set of points:

$$0 = 9 - x^2 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3 \Rightarrow (3,0), (-3,0)$$

If the given curve is $y=9-x^2$, then, the slope of the tangent line is the first derivative of the given curve at a particular point $P(\pm3,0)$:

$$f'(x)=-2x$$
$$f'(3)=-2(3)=-6=m_2 \quad \text{at a point } P(+3,0)$$
$$f'(-3)=-2(-3)=6=m_1 \quad \text{at a point } P(-3,0)$$

The tangent lines (27) on the given curve at the particular points are:

$$(y-0)=-6(x-3),\ m=-6,\ P(3,0)$$
$$6x+y-18=0$$
$$(y-0)=6(x+3),\ m=6,\ P(-3,0)$$
$$6x-y+18=0$$

The normal lines (28) on the given curve at the particular points are:

$$(y-0)=\frac{-1}{-6}(x-3),\ m=-6,\ P(3,0)$$
$$6y=x-3$$
$$x-6y-3=0$$
$$(y-0)=\frac{-1}{6}(x+3),\ m=6,\ P(-3,0)$$
$$6y=-x-3$$
$$x+6y+3=0$$

1. **The equation of a tangent at a point $P(x_0,y_0)$ is $(y-y_0)=m(x-x_0)$.**
2. **The equation of a normal at a point $P(x_0,y_0)$ is $(y-y_0)=\frac{-1}{m}(x-x_0)$.**

iii) *The angle of intersection of the two curves*

If m_1 is the slope of the first curve and m_2 is the slope of the second curve, then the angle of intersection in between these two curves at a point of intersection is the angle in between their tangents at that point. This angle takes the notation:

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2} \qquad (29)$$

The proof is available in Unit-6

Example 3.3.3: [Angle of Intersection]: Find the angle of intersection in between the curves $y = x^3 - 2x + 1$ and $y = x^2 + 1$ at the point of intersection (2, 5).

Solution: The required angle of intersection in between the given two curves is:

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \qquad (30)$$

For point of intersection, solve the system of nonlinear equations for the unknowns x and y:

$$y = x^3 - 2x + 1, \; y = x^2 + 1 \qquad (31)$$

Using first equation of the nonlinear system (31) in second equation to obtain:

$$x^3 - 2x + 1 = x^2 + 1$$

$$x^3 - x^2 - 2x = 0$$

$$x(x^2 - x - 2) = 0 \;\Rightarrow\; x = 0, -1, 2$$

The set of x values is used in first equation of the nonlinear system (31) to obtain a set of y values:

Put $x = 0$ to obtain $y = x^3 - 2x + 1 = 1$

Put $x = -1$ to obtain $y = x^3 - 2x + 1 = -1 + 2 + 1 = 2$

Put $x = 2$ to obtain $y = x^3 - 2x + 1 = 8 - 4 + 1 = 5$

This process developed a set of points of intersection: *(0,1), (-1,2), (2,5)*

The slope of the first curve at a point (2, 5) is:

$$\frac{dy}{dx} = 3x^2 - 2$$

$$\left(\frac{dy}{dx}\right)_{2,5} = 3(2)^2 - 2 = 10 = m_1, \text{ say}$$

The slope of the second curve at a point (2, 5) is:

$$\frac{dy}{dx} = 2x$$

$$\left(\frac{dy}{dx}\right)_{2,5} = 2(2) = 4 = m_2, \text{ say}$$

The slopes m_1 and m_2 are used in (30) to obtain the angle of intersection in between the given two curves:

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{10-4}{1+(10)(4)}$$

$$= \frac{6}{41}$$

$$\theta = \tan^{-1}\frac{6}{41} = 0.1453$$

Exercise 3.2

1. In each case, find the equation of the tangent line to the curve at the indicated value of x:

a. $y = \sqrt{x+1}, \quad x = 3$ b. $y = \sin(2x+\pi), \quad x = 0.$

c. $y = x^2 e^{-x}, \quad x = 1$ d. $y = \dfrac{2x+1}{x+2}, \quad x = 2$

e. $y = \dfrac{x}{x^2+1}, \quad x = 1$ f. $y = 3\sin x - \cos x, \quad x = \pi$

g. $y = 2\ln x, \quad x = e$ h. $y = 3e^x + e^{-x}, \quad x = 0$

2. In each case, find the equation of normal to the curve at the indicated value of x:

a. $y = xe^x, \quad x = 1$ b. $y = 2\sin 3x, \quad x = \pi$

c. $y = 2\ln x, \quad x = 1$ d. $y = (2x+1)^6, \quad x = 0$

e. $y = \dfrac{e^x + 1}{x}, \quad x = 1$ f. $y = \cos(x-\pi), \quad x = \pi/2$

g. $y = x^3 \ln x, \quad x = 1$ h. $y = \sqrt{x^2+1}, \quad x = 2$

3. a. Find an equation of the tangent line to the curve $x^2 + y^2 = 13$ at (–2, 3).

b. Find an equation of the tangent line to the curve $\sin(x-y) = xy$ at $(0, \pi)$.

c. Find an equation of the normal line to the curve $x^2 + 2xy = y^3$ at (1, –1).

4. a. Show that the first four terms in the Taylor series expansion of $f(x) = \tan x$ about $x = \pi/4$: are:

$$1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

b. Show that the first four terms in the Taylor series expansion of $f(x)=\sqrt{x}$ about $x=4$ are:

$$2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3$$

c. Show that the first four terms in the Taylor series expansion of $f(x)=x+e^x$ about $x=1$ are:

$$(1+e)x+e\left[\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\frac{(x-1)^4}{4!}+\dots\right]$$

5. Find the Maclaurin series expansion for the following functions:

a. $f(x)=\frac{1}{1+x}$ b. $f(x)=\sin^2 x$

c. $f(x)=\cos hx$ d. $f(x)=\ln(1-4x)$

6. a. Use the Maclaurin series for e^x to show that the sum of the infinite series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$ is e.

b. Use part (a) to find out the value of e that must be accurate to 4 decimal places.

7. Find the angle of intersection between the following curves:

a. $x^2-y^2=a^2,\ x^2+y^2=a^2\sqrt{2}$ b. $y^2=ax,\ x^3+y^3=3axy$

3.4 Maxima and Minima

Always the maximum and minimum values of a function can be read from its graphical view. For a quadratic function (whose graph is parabola), the maximum or minimum values can be determined without graphing by finding the vertex algebraically. For functions whose graphs are not known, other techniques are needed. In this unit, we shall see how to use derivatives to determine the maximum and minimum values of a function as well as the intervals where the function is increasing or decreasing.

i) *Definitions of increasing and decreasing functions*

Suppose an ecologist has determined the size of a population of a certain species as a function f of time t (months). If it turns out that the population is increasing until the end of the first year and decreasing thereafter. It is reasonable to

expect the population to be maximized at time t=12 and for the population curve to have a high point at $t = 12$ as shown in the figure (3.3).

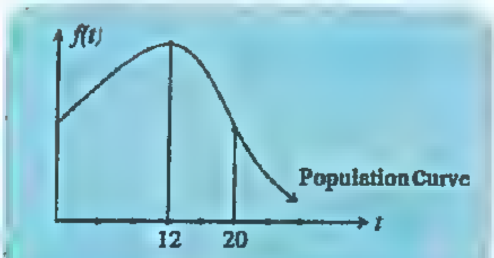


Fig. 3.3

If the graph of a function $f(t)$, such as this population curve, is rising throughout the interval $0<t<12$, then we say that $f(t)$ is strictly increasing on that interval. Similarly, the graph of the function in figure (3.3) is strictly decreasing on the interval $12 < t < 20$. These terms are defined more formally in the figure (3.4):

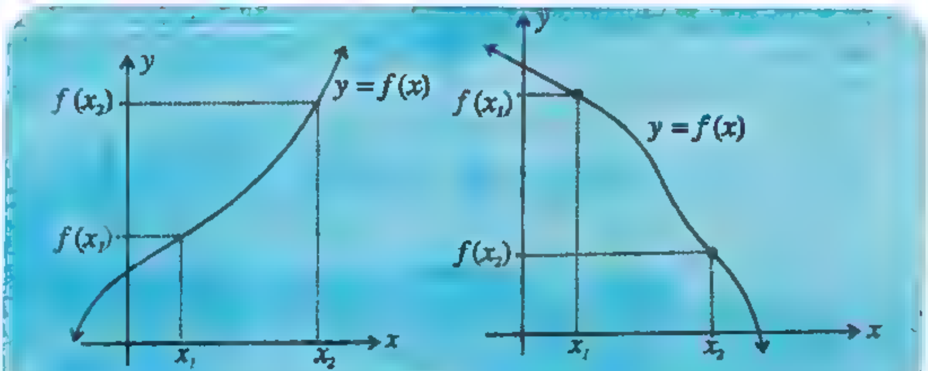


Fig.3.4

- The function $f(x)$ is **strictly increasing** on an interval (a, b), if $f(x_1) < f(x_2)$, whenever $x_1 < x_2$ for x_1 and x_2 in (a, b).
- The function $f(x)$ is **strictly decreasing** on an interval (a, b), if $f(x_1) > f(x_2)$, whenever $x_1 < x_2$ for x_1 and x_2 in (a, b).

Example 3.4.1:[Increasing and Decreasing Function]: Find the intervals at which the function $f(x) = x^2$ is increasing or decreasing.

Solution: The function $f(x) = x^2$ is a parabola passing through the origin. Take any two points x_1 and x_2 in the interval (a, b) for which:

$$f(x_2)-f(x_1)=x_2^2-x_1^2=(x_2-x_1)(x_2+x_1)$$

If $x_1, x_2 \in (0,\infty)$ with condition $x_2 > x_1$, then the function f(x) is increasing in the interval $(0,\infty)$:

$$f(x_2)-f(x_1)>0$$

$$f(x_2)>f(x_1), \quad both\ (x_2-x_1)\, and\ (x_2+x_1)\, are\ +ve, when\ x_2 > x_1$$

If $x_1, x_2 \in (-\infty,0)$ with condition $x_2 > x_1$, then the function f(x) is decreasing in the interval $(-\infty,0)$:

$$f(x_2)-f(x_1)<0$$

$$f(x_2)<f(x_1), \quad (x_2-x_1)\ is\ +ve\ \ while\ \ (x_2+x_1)\ is -ve, when\ x_2 > x_1$$

ii) *Proof of increasing and decreasing functions*

A function f(x) is said to be strictly **monotonic** on an interval (a, b), if it is either strictly increasing on (a, b) or strictly decreasing on (a, b) for all x. Monotonic behavior is closely related to the sign of the derivative $f'(x)$. In particular, if the graph of a function has tangent lines with only positive slopes on (a, b), then the graph will be tilted upward and f(x) will be increasing on (a, b). The slope of the tangent at each point on the graph is definitely measured by the derivative $f'(x)$. It is reasonable to expect f(x) to be increasing on intervals where $f'(x)>0$ and decreasing on an interval where $f'(x)<0$.

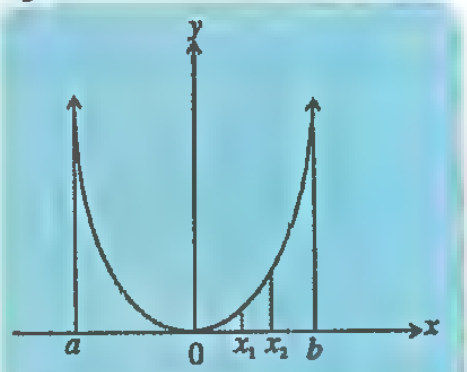


Fig. 3.5

These observations are established formally in Theorem 3.1:

Theorem 3.1: [Increasing & Decreasing Function Theorem]: If *f (x)* is differentiable on the open interval *(a, b)*, then the function

1. **f(x) is strictly increasing** on *(a, b)* if $f'(x)>0$ for $a<x<b$.
2. **f(x) is strictly decreasing** on *(a, b)* if $f'(x)<0$ for $a<x<b$.

This means that a differential function f(x)

- is increasing on an interval (a,b) if the tangent lines to its graph at a point (x,f(x)) makes positive slope $(f'(x) > 0)$.
- is decreasing on an interval (a,b) if the tangent line to its graph at a point (x,f(x)) makes negative slope $(f'(x) < 0)$.
- is neither increasing nor decreasing on an interval (a,b) if the tangent line to its graph at a point (x,f(x)) makes zero slope $(f'(x) = 0)$.

The proper proof is beyond of this text, since the proof is based on mean value theorem.

Example 3.4.2:[Increasing and Decreasing Function]: Determine the values of x at which the function $f(x) = -x^2 - 10x - 5$ is increasing or decreasing. Also find the point at which the given function is neither increasing nor decreasing.

Solution

For graphical view, the given function through completing square

$$f(x) = x^2 + 2x - 3 = x^2 + 2x + 1 - 1 - 3 = (x+1)^2 - 4$$

is compared with the general equation of parabola $f(x) = a(x-h)^2 + k$ to obtain a parabola with vertex (-1,-4) that opens upward (a = 1 is positive). The graph of a parabola through the points (-4,5) and (2,5) is shown in the figure (3.6):

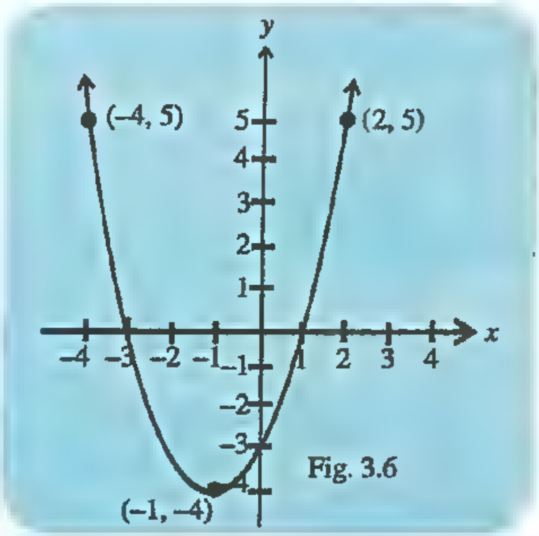


Fig. 3.6

The derivative of a given function with respect to x is the slope of the parabola:

$f'(x) = 2x + 2$

If the slope of parabola is $f'(x) > 0$ (positive), then it gives

$$\begin{aligned} f'(x) &> 0 \\ 2x + 2 &> 0 \\ 2x &> -2 \\ x &> -1 \end{aligned}$$

This shows that the given function f(x) is increasing in the interval $(-1, \infty)$.

If the slope of parabola is $f'(x) < 0$ (negative), then it gives

$$\begin{aligned} f'(x) &< 0 \\ 2x + 2 &< 0 \\ 2x &< -2 \\ x &< -1 \end{aligned}$$

This shows that the given function f(x) is decreasing in the interval $(-\infty, -1)$.

If the slope of parabola is $f'(x) = 0$ (zero), then it gives

$$f'(x) = 0 \Rightarrow 2x + 2 = 0 \Rightarrow 2x = -2 \Rightarrow x = -1$$

This shows that the given function f(x) is neither increasing nor decreasing at a vertex $(-1, -4)$.

iii) *Examine a given function for extreme values*

Typically the extrema of a continuous function occur either at endpoints of the interval or at points where the graph has a "peak" or a "valley" (points where the graph is higher or lower than all nearby points). For example, the function $f(x)$ in figure (3.7) has "peaks" at B and D and "valleys" at C and E. Peaks and valleys are what we call the **relative extrema.**

The exact location of a relative maximum or minimum rather than a graphic's approximation can normally be found by using derivatives. The concept developed is as under:

Let f (x) be a function as a roller coaster track with a roller coaster car moving from left to right along the graph in the figure (3.7). As the car moves up towards a peak, its floor tilts upward. At the instant the car reaches the peak, its floor is level, but then it begins to tilt downward as the car rolls down toward a valley.

At any point along the graph, the floor of the car (a straight-line segment in the figure) represents the tangent line to the graph at that point. Using this analogy, we see that as the car passes through the peaks and valleys at B and C, the tangent

line is horizontal and has slope 0. At peak D and valley E, however, a real roller coaster car would have troubled. It would fly off the track at peak D and be unable to make the 90^{0} change of direction at valley E. Here is no tangent line at D or E, because of the sharp corners.

Thus, the points where a peak or a valley occurs have this property: the tangent line is horizontal and has slope 0 there or no tangent line is defined there. The slope of the tangent line to the graph of the function f(x) at a point P(x, f(x)) is the value of the derivative $f'(x)$.

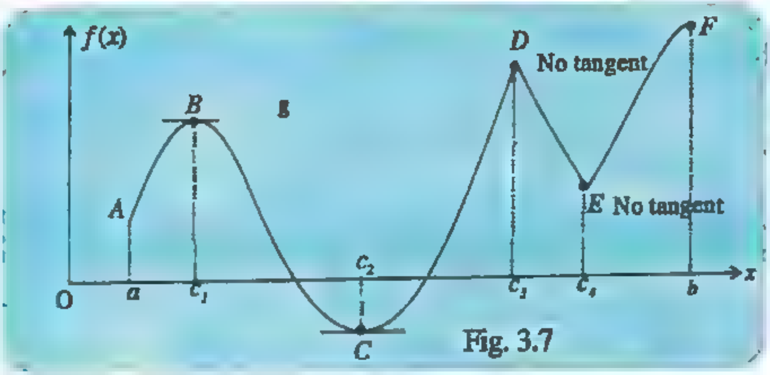


Fig. 3.7

Relative Maximum and Relative Minimum: The function $f(x)$ is said to have a **relative maximum** at a number c if $f(c) \geq f(x)$ for all x in an open interval containing c. Also, f (x) is said to have a **relative minimum** at a number d if $f(d) \leq f(x)$ for all x in an open interval containing d. In general, the relative maxima and relative minima are called **relative extrema.**

Critical Values and Critical Point: Suppose $f(x)$ is defined at a number c and either $f'(c)=0$ or $f'(c)$ does not exist. Then the number c is called a **critical value** of $f(x)$ and the point $P(c, f(c))$ on the graph of $f(x)$ is called a **critical point.**

Note that if $f(c)$ is not defined, then c cannot be a critical value. If there is a relative maximum at c, then the functional value $f(c)$ at that point is the maximum value. Similarly, if there is a relative minimum at c, then the functional value f(c) at that point is the minimum value.

Example 3.4.3: [Critical Values]: Find the critical values for the following functions:

a. $f(x)=4x^3-5x^2-8x+20$ b. $f(x)=\frac{x^2}{x-2}$ c. $f(x)=12x^{1/2}-2x^{3/2}$

Solution:

a. The first derivative of the given function is:

$f'(x)=12x^2-10x-8$

$f'(x)=12x^2-10x-8$ is defined for all values of x. Set $f'(x)=0$ to obtain the critical values:

$$f'(x)=12x^2-10x-8=0 \Rightarrow 2(3x-4)(2x+1)=0 \Rightarrow x=\frac{4}{3},\frac{-1}{2}$$

b. The first derivative of the given function is:

$$f'(x)=\frac{x(x-4)}{(x-2)^2}$$

The derivative is not defined at x=2, also the orignal function $f(x)$ is not defined at x =2. So x =2 is not a critical value. Set $f'(x)=0$ to obtain the other critical values:

$$f'(x)=\frac{x(x-4)}{(x-2)^2}=0 \Rightarrow x(x-4)=0 \Rightarrow x=0,4$$

c. The first derivative of the given function is:

$f'(x)=6x^{-1/2}-3x^{1/2}$

The derivative is not defined at x =0, but the original function f (x) at x=0 is $f(0)=12(0)^{1/2}-2(0)^{3/2}=0$, which is defined. So x=0 is a critical value. For other critical values, set $f'(x)=0$ to obtain:

$$f'(x)=6x^{\frac{-1}{2}}-3x^{\frac{1}{2}}=0 \Rightarrow 3x^{\frac{-1}{2}}(2-x)=0 \Rightarrow 2-x=0 \Rightarrow x=2$$

Thus, the critical values are x=0, 2.

Theorem 3.2: [Critical Value Theorem]: If a continuous function $f(x)$ has a relative extremum at c, then c must be a critical value of $f(x)$.

Example 3.4.4: [Determination of intervals on which a function is increasing or decreasing]: The function $f(x)$ is defined by $f(x)=x^3-3x^2-9x+1$. Determine the intervals at which the function $f(x)$ is strictly increasing or decreasing.

Solution:

First, we need to find out the derivative of the given function, which is:

$$f'(x)=3x^2-6x-9$$

For critical values, set $f'(x)=0$ to obtain:

$$3x^2-6x-9=0 \Rightarrow 3(x+1)(x-3)=0 \Rightarrow x=-1,3$$

These critical values divide the x– axis into three parts, as shown in the fig.3.8. Next, we select a typical number from each of these intervals. For example, we select –2, 0 and 4, evaluate the derivative at these values and mark each interval as increasing or decreasing, according to whether the derivative is positive or negative respectively. This is shown in Fig.3.8

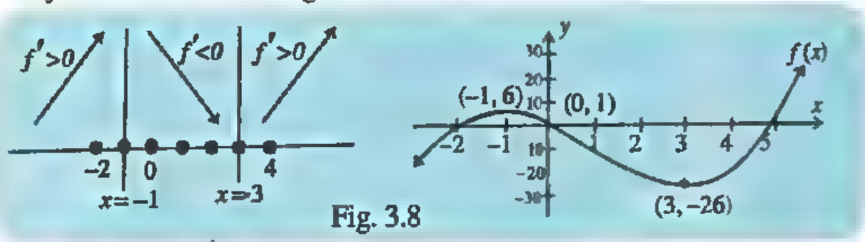


Fig. 3.8

Thus, the function $f(x)$ increases in the intervals for $x< -1$ and $x > 3$, but decreases in the interval $-1<x<3$.

Example 3.4.5: Draw the function $f(x)=x^3-3x^2-9x+1$ and its derivative $f'(x)=3x^2-6x-9$. Use these graphs to tell about the following questions:

a. When $f'(x)$ is positive, what does that mean in terms of the graph of $f(x)$?

b. When the graph of $f(x)$ is decreasing, what does that mean in terms of the graph of $f'(x)$?

Solution: The graphs of $f(x)=x^3-3x^2-9x+1$ and $f'(x)=3x^2-6x-9$ are shown in figure (3.9):

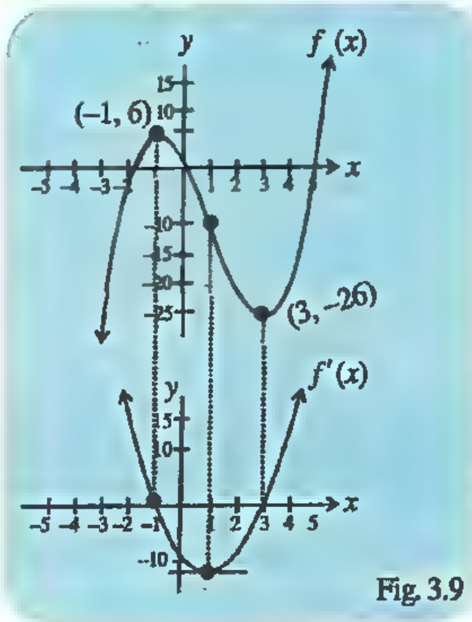


Fig. 3.9

These graphs develop the idea that the critical values of $f(x)$ are always intercepts for the graph of $f'(x)=3x^2-6x-9$:

- If $f'(x)$ is positive, then $f(x)$ is increasing.
- If $f'(x)$ is negative, then $f(x)$ is decreasing.

iv) The First–derivative rule to find the extreme values of a function at a point

The first derivative of a function can be used to determine whether the function is increasing or decreasing on a given interval. We shall use this information to develop a procedure called the **first derivative test** for classifying a given point as a relative maximum, a relative minimum, or neither.

The steps involved in first–derivative test for relative extrema are the following:

1. Find all critical values of $f(x)$. That is, find all numbers c such that $f(c)$ is defined and either $f'(c)=0$ or $f'(c)$ does not exist.
2. The point $(c, f(c))$ is a **relative maximum** if $f'(x)>0$ (rising) for all x in an open interval (a, c) to the left of c, and $f'(x)<0$ (falling) for all x in an open interval (c, b) to the right of c.
3. The point $(c, f(c))$ is a **relative minimum** if $f'(x)<0$ (falling) for all x in an interval (a, c) to the left of c, and $f'(x)>0$ (rising) for all x in an open interval (c, b) to the right of c.
4. The point $(c, f(c))$ is not an **extremum** if the derivative $f'(x)$ has the same sign in open intervals (a, c) and (c, b) on both sides of c.

In light of first-derivative test, the function $f(x)=x^3-3x^2-9x+1$ (example 3.4.4) has the critical values –1 and 3. The function $f(x)$ is increasing when $x<-1$ and $x>3$ and decreasing when $-1<x<3$. The first derivative test tells us that there is a relative maximum of 6 at $x=-1$ and a relative minimum of –26 at $x=3$.

Example 3.4.6: [Relative Extrema]: Examine the function $f(x)=2x^3+3x^2-12x-5$ for the relative extrema using first–derivative test.

Solution:

The first derivative of $f(x)$ is:

$$f'(x)=6x^2+6x-12=6(x+2)(x-1)$$

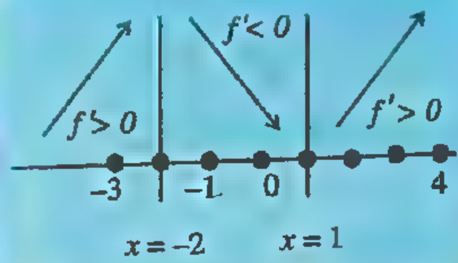


Fig. 3.10

Set $f'(x) = 0$ to obtain the critical values:

$$f'(x) = 6x^2 + 6x - 12 = 0$$
$$= 6(x+2)(x-1) = 0 \Rightarrow x = -2, 1$$

To test the critical values −2, 1, we can use the test values −3, 0 and 2. Many other choices of the test values are also possible, but we try to select numbers that will make the computations easy. This is shown in the figure (3.10).

The test values −3 and 0 are used for the critical value $x = -2$ to obtain:

$$f'(-3) = 6(-3+2)(-3-1) = 24 > 0 \ (positive)$$
$$f'(0) = 6(0+2)(0-1) = -12 < 0 \ (negative)$$

The value of the derivative is positive (rising) to the left of −2 and negative (falling) to the right of −2. Thus, $x = -2$ leads a **relative maximum point** $f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) - 5 = -16 + 12 + 24 - 5 = 15$.

The test values 0 and 2 are used for the critical value $x=1$ to obtain:

$$f'(0) = 6(0+2)(0-1) = -12 \ (negative)$$
$$f'(2) = 6(2+2)(2-1) = 24 \ (positive)$$

The value of the derivative is negative (falling) to the left of 1 and positive (rising) to the right of 6. Thus, $x=1$ leads a **relative minimum point.** $f(1) = 2(1) + 3(1) - 12(1) - 5 = -12$

Thus, the arrow pattern in the figure suggests that the graph of $f(x)$ has a relative maximum at (−2, 15) and a relative minimum at (1,−12).

v) *The Second-derivative rule to find the extreme values of a function at a point*

Concavity: A portion of a graph that is cupped upward is called **concave up**, and a portion that is cupped downward is **concave down.** Figure (3.11) shows a graph that is concave up between A and C and concave down between C and E. At various points on the graph, the slope is indicated by "arrows," and we observe that the slope increases from A to C and decreases from C to E. This is not accidentally, but under the rule **"the slope of a graph increases on an interval where the graph is concave up and decreases, where the graph is concave down".**

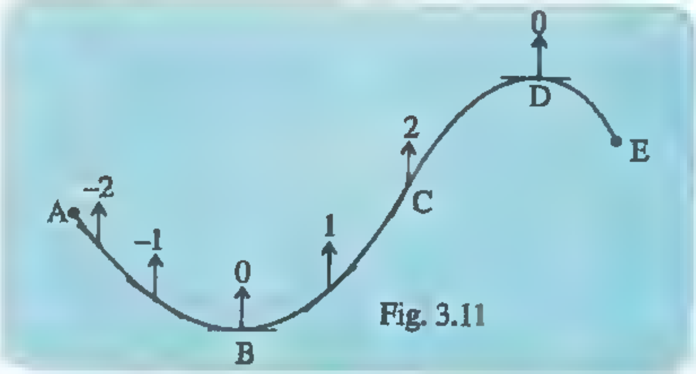


Fig. 3.11

Conversely, a graph will be concave up on any interval where the slope is increasing and concave down where the slope is decreasing. The slope is actually the derivative $f'(x)$ of a given function f(x). It is reasonable to expect that the graph of a given function f(x) is concave up when the derivative $f'(x)$ is strictly increasing and concave down when the derivative $f'(x)$ is strictly decreasing. Through theorem (3.1), the graph of f(x) is concave up when the second derivative $f''(x)$ satisfies $f''(x)>0$. The graph is of course concave down when $f''(x)<0$. This observation is used to define the concavity.

Definition 3.4.1: [Concavity]: The graph of a function f *(x)* is **concave upward** on an open interval *(a, b)* where $f''(x)>0$ and it is **concave downward** where $f''(x)<0$.

Inflection Point: In general, an **inflection point** is a point $P(c, f(c))$ on the graph of a function $y=f(x)$ at which the concavity changes from upward to downward or from downward to upward. In order to change the concavity at a point, the derivative $f''(x)$ must changes sign at that point.

Theorem 3.4.2: [Inflection Points]: If *y=f (x)* is continuous on *(a, b)* and has an inflection point at *x=c*, then either $f''(c)=0$ or $f''(c)$ does not exist.

Note that the inflection points can occur only at partition numbers of $f''(x)$, but not every partition number of $f''(x)$ produces an inflection point.

A partition number c for $f''(x)$ produces an inflection point for the graph of $f(x)$ only if:

- $f''(x)$ changes sign at partition number c.
- partition number c is in the domain of $f(x)$.

This is summarized in a box:

1. A point $P(c, f(c))$ on the graph of a differential function $y=f(x)$ where the concavity changes is called a point of inflection.
2. If a function has a point of inflection $P(c, f(c))$ at a partition c and it is possible to differentiate the function twice, then $f''(c)=0$.

Example 3.4.7: [Concavity]: Test the function $f(x)=x^3-6x^2+9x+1$ for concavity? Is there any inflection point?

Solution: The first and second derivatives of the given function are the following:

$$f'(x)=3x^2-12x+9$$

$$f''(x)=6x-12=6(x-2), \quad \textit{partition point is 2}$$

If $x<2$, say, $x=1$, then $f''(1)=6(1-2)=-6<0$, so, the graph of $f(x)$ is concave down for $x<2$.

If $x>2$, say, $x=3$, then $f''(3)=6(3-2)=6>0$, so, the graph of $f(x)$ is concave up for $x>2$.

Thus, the graph of $f(x)$ has an inflection point at $x=2$. Put $x=2$ in the given function $f(x)=x^3-6x^2+9x+1$ to obtain an inflection point (2, 3). This is the point at which concavity changes. This is shown in the figure (3.12).

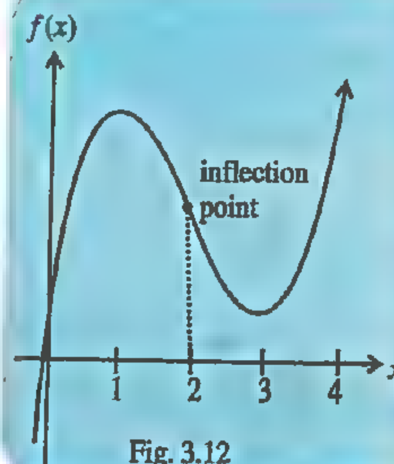
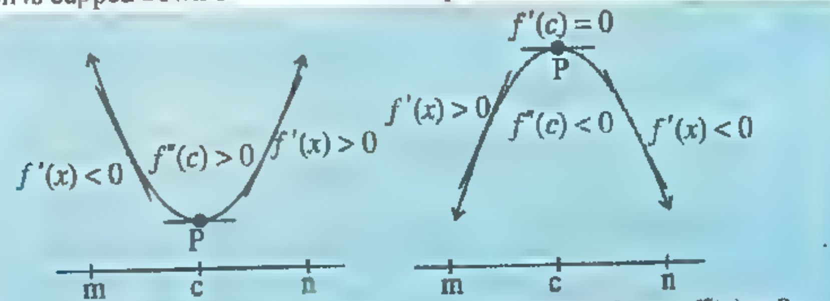


Fig. 3.12

The Second–Derivative Rule: It is often possible to classify a critical point P(c, f(c)) on the graph of f(x) by examining the sign of $f''(c)$. Specifically, if $f'(c)=0$ and $f''(c)>0$, then there is a horizontal tangent line at P and the graph of f(x) is concave up in the neighborhood of P. This means that the graph of f(x) is cupped upward from the horizontal

tangent at P and to expect P to be a relative minimum, as shown in figure (3.13(a)). Similarly, we expect P to be a relative maximum, if $f'(c)=0$ and $f''(c)<0$, because the graph is cupped down beneath the critical point P, as shown in Fig. 3.13(b)



(a) $f'(c)=0$ and $f''(c)>0$ implies $f(c)$ is a relative minimum

(b) $f'(c)=0$ and $f''(c)<0$ implies $f(c)$ is a relative maximum

Fig. 3.13

In other words,

- The point P($c, f(c)$) is said to be a relative maximum, if the slope $f'(c)$ of the tangent line from left to right along a curve through P, is decreasing from **positive to zero to negative** and the second derivative $f''(c)$ is negative.
- The point P($c, f(c)$) is said to be relative minimum, if the slope $f'(c)$ of the tangent line from left to right along a curve through P, is increasing from **negative to zero to positive** and the second derivative $f''(c)$ is positive.

These observations lead to the second–derivative test for relative extreme.

The Second Derivative Rule for Relative Extrema: Let $f(x)$ be a function such that $f'(c)=0$ and the second derivative exists on an open interval (a,b) containing c.

1. If $f''(c)>0$, then there is a **relative minimum** at x=c and the graph of $f(x)$ is **concave up** in the neighborhood of P($c, f(c)$).
2. If $f''(c)<0$, then there is a **relative maximum** at x=c and the graph of $f(x)$ is **concave down** in the neighborhood of P($c, f(c)$).
3. If $f''(c)=0$, then the second derivative test fails and gives no information.

vi) *Use of second-derivative rule to examine a given function for extreme values*

Example 3.4.8: Use the second-derivative test to determine whether each critical value of the function $f(x)=3x^5-5x^3+2$ corresponds to a relative maximum, a relative minimum, or neither.

Solution: The first and second derivatives of $f(x)$ are the following:

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x-1)(x+1),\ f''(x) = 60x^3 - 30x = 30x(2x^2-1)$$

Put $f'(x) = 15x^4 - 15x^2 = 15x^2(x-1)(x+1) = 0$ to obtain the critical values 0, 1 and –1.

The second derivative $f''(x)$ at a critical point x=0 is:

$$f''(0) = 30(0)(0-1) = 0$$

The critical value x=0 declares the failure of second derivative test.

The second derivative $f''(x)$at a critical point x=1 is:

$$f''(1) = 30 > 0$$

The critical value x=1 leads to a relative minimum of $f(1) = 3(1) - 5(1) + 2 = 0.$

The second derivative $f''(x)$at a critical point x= –1 is:

$$f''(-1) = -30 < 0$$

The critical value x = –1 gives a relative maximum of $f(-1) = -3 - 5(-1) + 2 = 4$

The second derivative test works only for those critical values c that make $f'(c) = 0$. This test does not work for those critical values c for which $f'(c)$ does not exist or that make $f''(c) = 0$. In both of these cases, use the first derivative test to proceed the process of relative extrema.

vii) Real-life problems related to extreme values

Example 3.4.9: A truck burns fuel at the rate of

$$C(x) = \frac{1}{200}\left(\frac{800 + x^2}{x}\right), \quad x > 0$$

gallons per mile when traveling x miles per hour on a straight level road. If fuel costs \$2 per gallon, find the speed that will produce the minimum total cost for a 1000 mile trip. Find the maximum total cost.

Solution: The total cost of the trip in dollars is the product of the(number of gallons per mile)(the number of miles)(the cost per gallon) that develops the rule:

$$C(x) = \frac{1}{200}\left(\frac{800 + x^2}{x}\right)(1000)(2) = \frac{8000 + 10x^2}{x}$$

The independent variable x represents speed, only positive values of x make sense here. Thus, the domain of C(x) is the open interval$(0, \infty)$and there are no end points to check.

The first and second derivatives of C(x) are the following:

$$C'(x) = \frac{10x^2 - 8000}{x^2}, \quad C''(x) = \frac{16000}{x^3}$$

Put $C'(x) = 0$ to obtain the critical values:

$$\frac{10x^2 - 8000}{x^2} = 0 \Rightarrow 10x^2 - 8000 = 0 \Rightarrow x^2 = 8000 \Rightarrow x = \pm 28.3\,mph$$

The only critical number in the domain is $x = 28.3$. The second derivative test at a critical value $x=28.3$ is:

$$C''(28.3)=\frac{16000}{(28.3)^3}=0.72>0$$

The second derivative test shows that the critical value $x=28.3$ leads to a minimum value. The minimum total cost is found by inserting $x=28.3$ in the cost function :

$$C(28.3)=\frac{8000+10(28.3)^2}{28.3}=565.69\text{dollars}$$

Exercise 3.3

1. a. What is the first–derivative test?
   b. What is the relationship between the graph of a function and the graph of its derivative?
   c. What is the second–derivative test?
   d. What is the relationship between concavity, points of inflection and the second–derivative test?
2. Find the critical values of the given functions and show where the function is increasing and where it is decreasing. Plot each critical point and label it as a relative maximum, a relative minimum, or neither.
   a. $f(x)=x^3+3x^2+1$ b. $f(x)=x^3+35x^2-125x-9.375$
3. Find the critical values of the following functions:
   a. $f(x)=2x^3-3x^2-72x+15$ b. $f(x)=\frac{1}{3}x^3-x^2-15x+6$
   c. $f(x)=6x^{2/3}-4x$ d. $f(x)=3x^{4/3}-12x^{1/3}$
4. Determine whether the given function has a relative maximum, a relative minimum, or neither at the given critical values for the following problems:
   a. $f(x)=(x^3-3x+1)^7=0$ at $x=1$; $x=-1$
   b. $f(x)=(x^4-4x+2)^5$ at $x=1$
   c. $f(x)=(x^2-4)^4(x^2-1)^3$ at $x=1$; $x=2$
   d. $f(x)=\sqrt[3]{x^3-48}$ at $x=4$
5. Find all relative extrema of the following functions:
   a. $f(x)=x^3-3x^2+1$ b. $f(x)=x^3+6x^2+9x+2$

6. a. Suppose $f(x)$ is a differential function with derivative

$f'(x) = (x-1)^2(x-2)(x-4)(x+5)^4$

Find all critical values of f(x) and determine whether each corresponds to a relative maximum, a relative minimum, or neither.

7. A company has found through experience that increasing its advertising also increases its sales up to a point. The company believes that the mathematical model connecting profit in hundreds of dollars $P(x)$ and expenditures on advertising in thousands of dollars x is:

$P(x) = 80 + 108x - x^3,\ 0 \le x \le 10$

a. Find the expenditure on advertising that leads to maximum profit.

b. Find the maximum profit.

8. The total profit $P(x)$ (in thousands of dollars) from the sale of x hundred thousands of automobile tyres is approximated by

$P(x) = -x^3 + 9x^2 + 120x - 400,\quad 3 \le x \le 15$

Find the number of hundred thousands of tyres that must be sold to maximize profit. Find the maximum profit.

9. The percent of concentration of a drug in the bloodstream x hours after the drug is administered is given by:

$$K(x) = \frac{4x}{3x^2 + 27}$$

a. On what time intervals is the concentration of the drug increasing?

b. On what intervals is it decreasing?

c. Find the time at which the concentration is a maximum.

d. Find the maximum concentration.

10. A diesel generator burns fuel at the rate of

$$G(x) = \frac{1}{48}\left(\frac{300}{x} + 2x\right)$$

gallons per hour when producing x thousand kilowatt hours of electricity. Suppose that fuel costs \$2.25 a gallon and find the value of x that leads to minimum total cost if the generator is operated for 32 hours. Find the minimum cost.

Glossary

- The second derivative of $y=f(x)$ can be written with any of the following notations

$$\frac{d^2y}{dx^2},\ y'',\ f''(x),\ D_x^2[f(x)]$$

The third derivative can be written in a similar way. For $n \geq 4$, the nth derivative is written as $f^{(n)}(x)$.

- The second derivative of parametric functions x(t) and y(t) can be found as follows:

$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx}$$

$$=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3},\quad Put\ \frac{dt}{dx}=\frac{1}{\left(\frac{dx}{dt}\right)}$$

- The popular notation for the Taylor series of order n is

$$f(x_0+h)=f(x_0)+f'(x_0)h+f''(x_0)\frac{h^2}{2!}+\ldots+f^n(x_0)\frac{h^n}{n!}+\ldots$$

- The popular notation for the Maclaurin's series of order n is:

$$f(x_0+h)=f(0)+f'(0)h+f''(0)\frac{h^2}{2!}+\ldots+f^n(0)\frac{h^n}{n!}+\ldots$$

- If two lines are parallel, then their slopes are equal.

ii. If two lines are perpendicular, then the product of their slopes equals -1.

iii. The tangent equation at a point $P(x_0, y_0)$ is $(y-y_0)=m(x-x_0)$.

iv. The normal equation at a point $P(x_0, y_0)$ is $(y-y_0)=\frac{-1}{m}(x-x_0)$.

Theorem: [Increasing & Decreasing Function Theorem]: If $f(x)$ is differentiable on the open interval (a, b), then the function $f(x)$ is

strictly increasing on (a, b) if $f'(x)>0$ for $a<x<b$.

strictly decreasing on (a, b) if $f'(x)<0$ for $a<x<b$.

- **Theorem: [Critical Value Theorem]:** If a continuous function $f(x)$ has a relative extremum at c, then c must be a critical value of $f(x)$.
- The graph of a function $f(x)$ is concave upward on an open interval (a, b), where $f''(x)>0$, and it is concave downward where $f''(x)<0$.
- **Theorem 4.3: [Inflection Points]:** If $y=f(x)$ is continuous on (a, b) and has an inflection point at $x=c$, then either $f''(c)=0$ or $f''(c)$ does not exist.

i. A point $P(c, f(c))$ on the graph of a differential function $y=f(x)$ where the concavity changes is called a **point of inflection.**

ii. If a function has a point of inflection $P(c, f(c))$ at a partition c and it is possible to differentiate the function twice, then $f''(c)=0$.

UNIT 4

DIFFERENTIATION OF VECTOR FUNCTIONS

This unit tells us, how to:

- define the scalar and vector functions with the understanding of its domain and range.
- define the limit and continuity of a vector-valued function.
- demonstrate the properties of limits of a vector-valued functions, such as the sum and difference, dot product, and cross product of two vector-valued functions, the product of a scalar and vector-valued functions .
- define the dferivative of a vector-valued function and to elaborate the Leibnitz result.
- prove the vector-valued differentiation formulae, such as the constant rule, linearity rule, scalar multiple rule, quotient rule, dot product rule, cross product rule, and chain rule.
- apply vector-valued differentiation to particle's position vector R(t) and its velocity V(t) and acceleration A(t).

4.1 Scalar and Vector Functions

The relationship of calculus and vector methods forms what is called **vector calculus.** The key to use vector calculus is the concept of a vector function. In this unit, we introduce such functions and examine some of their properties. We shall see that vector functions behave very much like scalar functions.

i) Definition of scalar and vector functions

Definition 4.1.1: [Scalar Function]: A function $f(x)$ is a rule which operates on an input x (x is any scalar quantity) and produces always just a single scalar output y. This gives a proper notation of a scalar function:

$$y=f(x) \qquad (1)$$

For example,

1. $C(x)=2x+2$ is a cost function that depends on x number of units of items. Here x is the input, y is the output and $C(x) = 2x + 2$ is the rule which operates on an input x to produce a single output quantity y.

   In response of $x=2$ items (2 is scalar), the cost (cost is also scalar) is:

   $C(2)=2(2)+2=6$ rupees.

   This function is then called a **scalar (single variable) function;** because it transforms one input x to produce just one output C.

2. $A(x, y) = xy$ is the area of rectangle that depends on length x and width y. Here x, and y are the two inputs and the rule $A(x, y) = xy$ which operates on two inputs x and y to produce a single output quantity A.

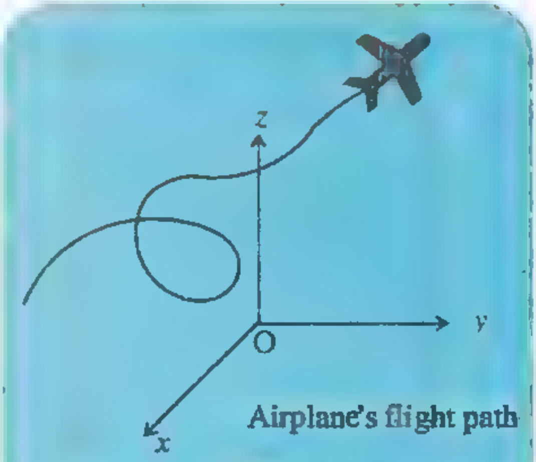
   In response of $x = 2$ and $y = 1$ (2 and 1 are scalars), the area is (is also a scalar) A (2, 1)=(2) (1)= 2 square units

   This function is then called a **scalar (double variables) function,** because it transforms two inputs x and y to produce just one output A.

   This idea can easily be extended to define a scalar multivariate function.

The uniqueness of scalar function is to transform **scalar quantities in a single scalar quantity.** Is there any rule that will transform **scalar quantities in a vector quantity**? Yes, the rule is the vector functions. Vector functions are used to study curves in the plane and space.

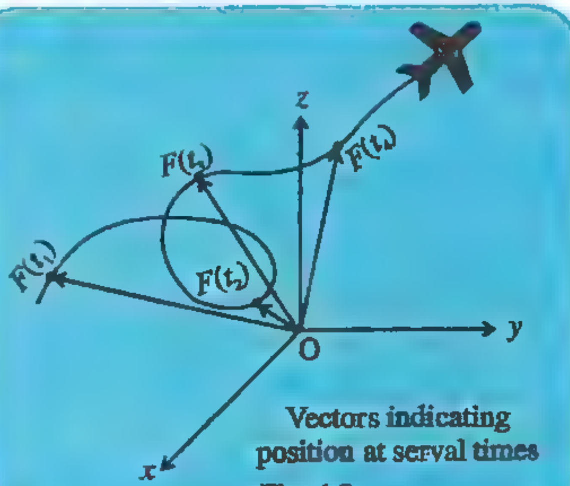
Vector Functions: Consider the circuitous path of the airplane. How could you describe its location at any given time? In this sense, pilots view their cockpit as the origin, the point from which measurements should be made. Further, an airplane's velocity determines a specific orientation of space, in terms of which relative directions such as "behind" and "above are defined. This is shown in the Fig. 4.1



Airplane's flight path

Fig. 4.1

You might consider the circuitous path of the airplane using a point $P(x, y, z)$ in 3-space, but it turns out to be more convenient if we describe its location at any given time by the endpoint of a vector in three dimensional space. We call such a vector a **position vector**. You should realize that the circuitous path of the airplane needs a vector for every time in three dimensional space.



Vectors indicating position at serval times
Fig. 4.2

Figure (4.2) indicates the location of the plane through position vectors $F(t_1)$, $F(t_2)$,... at a number of times $t_1, t_2, ...$ Notice that the function F(t) for time $t = t_1, t = t_2, ...$ would do the job of vector functions $F(t_1)$, $F(t_2)$,.... in 3-space nicely.

Definition 4.1.2: [Vector Functions]: A vector function **F** is a mapping from its domain D (a set of real numbers) to its range R (a set of three dimensional vectors; V_3) so that for each time t in D, F(t) = v for only one vector v in V_3.

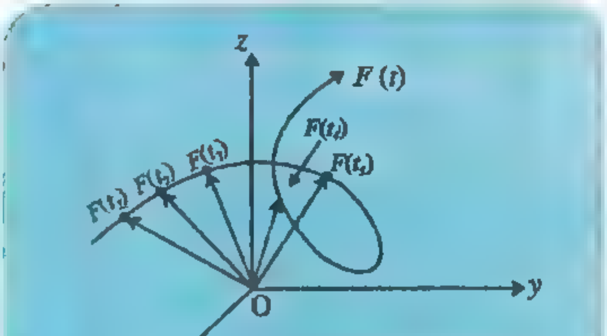
The proper notation of a vector function is:

$$F(t) = (x(t), y(t)) = \left(f_1(t), f_2(t)\right) = f_1(t)i + f_2(t)j, \quad 2-space \qquad (2)$$

$$F(t) = \left(x(t), y(t), z(t)\right) = \left(f_1(t), f_2(t), f_3(t)\right) = f_1(t)i + f_2(t)j + f_3(t)k, \quad 3-space \qquad (3)$$

Here $f_1(t), f_2(t)$ and $f_3(t)$ are scalar functions of the real number t defined on the domain D. In this context $f_1(t), f_2(t)$ and $f_3(t)$ are called the components of F (t) in response of independent variable *t*.

Let F (t) be a vector function. If the initial point of the vector F (t) is at the origin, then the graph of a vector F(t) is the curve traced out by the terminal point of the position vector F(t) as t varies over the domain set D. This is shown in the figure (4.3):



The graph of the vector function $F(t)$

Fig. 4.3

Example 4.1.1:[Sketch of Vector Function]: Sketch the curve traced out by the terminal point of the two-dimensional vector function:

$$F(t)=(t+1)i+(t^2-2)j \qquad (4)$$

Solution: The graph is the collection of all points (x, y) with $x=t+1,\ y=t^2-2$ for different values of t. The vector function F(t) is used for t = 0, 2, -2 to obtain the position vectors:

$t=0 \quad \Rightarrow \quad F(0)=(0+1)i+(0-2)j=i-2j=(1,-2)$

$t=2 \quad \Rightarrow \quad F(2)=(2+1)i+(4-2)j=3i+2j=(3,2)$

$t=-2 \quad \Rightarrow \quad F(-2)=(-2+1)i+(4-2)j=-i+2j=(-1,2)$

We plot these position vectors in figure (4.4a). The terminal points of all position vectors F(0), F(2), F(−2) lie on the curve described parametrically by,

$x=t+1,\ y=t^2-2,\ t\in R$

For elimination of parameter t, we set t=x-1 in $y=t^2-2$ to obtain the curve in xy-plane:

$y=t^2-2=(x-1)^2-2$

(5)

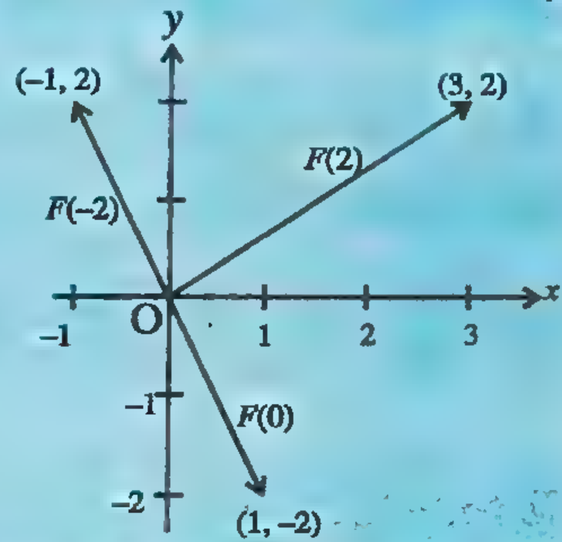


Figure 4.4a: Some value of $F(t)=(t+1)i+(t^2-2)j$

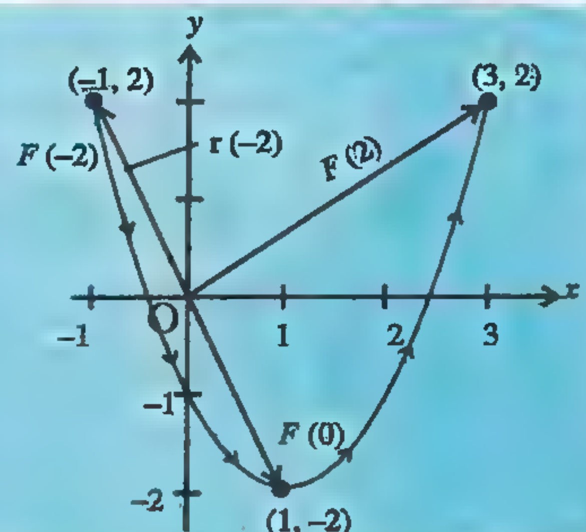


Figure 4.4b: Graph of $F(t)=(t+1)i+(t^2-2)j$

Notice that the graph of this vector function is a parabola opening up, with vertex at the point (1,-2), as seen in figure (4.4b). The small arrows marked on the graph indicate the orientation, that is, the direction of increasing values of t. If the curve describes the path of an object, then the orientation indicates the direction in which the object traverses the path. In this case, we can easily

determine the orientation from the parametric representation of the curve. Since $x = t + 1$, we observe that x increases as t increases.

ii) *Domain and range of a vector function*

Definition 4.1.3:[Domain and Range]: The set of all t values used as input in $F(t)$ is called the **domain** of a vector-valued function $F(t)$ and the set of $F(t)$ values that the vector function $F(t)$ takes as t varies, is called the **range** of a vector function $F(t)$.

Example 4.1.2: [Domain and Range]: Find the domain for the following vector function $F(t) = 2ti - 3tj + t^{-1}k$.

Solution:

The vector function is:

$$F(t) = \left(f_1(t), f_2(t), f_3(t)\right) = 2ti - 3tj + t^{-1}k$$

The function $f_1(t) = 2t$ is defined for all t; $f_2(t) = 3t$ is defined for all values of t; $f_3(t) = t^{-1}$ is defined for all values of t except $t = 0$. Thus, the domain of the function F(t) is R – {0}

- *Operations with vector functions*

It follows from the definition of vector operations that vector functions can be added, subtracted, multiplied by a scalar function, and multiplied together.

Definition 4.1.4:[Vector Operations]: If F and G are vector functions of the real variable t, and h(t) is any scalar function, then F+G, F-G and $F \times G$ are vector functions, and F.G is a scalar function. These operations are:

Vector Functions:

- **Addition:** (F+G)(t)=F(t)+G(t)
- **Subtraction:** (F–G)(t)=F(t)+G(t)
- **Scalar product:** (h F)(t)=h(t) F(t)
- **Cross product:** $\left(F \times G\right)t = F(t) \times G(t)$

Scalar Function:

Dot product: (F G)(t)=F (t). G (t)

Example 4.1.3: [Vector Function Operations]: Let $F(t) = t^2 i + tj - \sin t\, k$ and $G(t) = ti + t^{-1}j + 5k$ are the two vector-valued functions. Find the following vector operations:

a. (F+G)(t) b. $\left(e^t F\right)(t)$ c. $\left(F \times G\right)(t)$ d. $\left(F.G\right)(t)$

Solution:

a. If $F(t)=t^2i+tj-\sin t\,k$ 1and $G(t)=t\,i+t^{-1}j+5k$, then

$$(F+G)(t)=F(t)+G(t)$$

$$=(t^2i+tj-\sin t\,k)+\left(ti+t^{-1}j+5k\right)=(t^2+t)i+(t+t^{-1})j+(5-\sin t)k$$

b. $\left(e^tF\right)(t)=e^tF(t)=e^t\left(t^2i+tj-\sin t\,k\right)=e^tt^2i+e^ttj-e^t\sin t\,k$

c. $\left(F\times G\right)(t)=F(t)\times G(t)$

$$=\left(t^2i+tj-\sin tk\right)\times\left(ti+t^{-1}j+5k\right)$$

$$=\begin{vmatrix} i & j & k \\ t^2 & t & -\sin t \\ t & t^{-1} & 5 \end{vmatrix}$$

$$=\left(5t+\frac{\sin t}{t}\right)i-(5t^2+t\sin t)\,j+(t-t^2)k$$

d. $(F.G)(t)=F(t).G(t)$

$$=\left(t^2i+tj-\sin t\,k\right).\left(ti+t^{-1}j+5k\right)$$

$$=t^3+1-5\sin t,\quad i.i=j.j=k.k=1$$

4.2 Limits and Continuity

For the most part, vector limits behave like scalar limits. The proper definition of the limit of a vector function is given below.

i) *Limit of a vector function*

Definition 4.2.1: [Limit of a Vector Function]: If all the components $f_1(t), f_2(t), f_3(t)$ of a vector function

$$F(t)=\left(f_1(t), f_2(t), f_3(t)\right)=f_1(t)i+f_2(t)j+f_3(t)k$$

have finite limits as $t\to t_0$, where t_0 is any number or $\pm\infty$, then the limit of F(t) as $t\to t_0$ is the vector:

$$\lim_{t\to t_0} F(t) = \lim_{t\to t_0}\left[f_1(t)i + f_2(t)j + f_3(t)k\right]$$
$$= \left[\lim_{t\to t_0} f_1(t)\right]i + \left[\lim_{t\to t_0} f_2(t)\right]j + \left[\lim_{t\to t_0} f_3(t)\right]k$$

Example 4.2.1:[Limit of a Vector Function]: Find $\lim_{t\to 2} F(t)$, when the vector function is $F(t) = (t^2-3)i + e^t j + \sin \pi t k$.

Solution:

$$\lim_{t\to 2} F(t) = \lim_{t\to 2}\left[f_1(t)i + f_2(t)j + f_3(t)k\right]$$
$$= \left[\lim_{t\to 2}(t^2-3)\right]i + \left[\lim_{t\to 2}(e^t)\right]j + \left[\lim_{t\to 2}\sin \pi t\right]k$$
$$= (4-3)i + e^2 j + \sin 2\pi k = i + e^2 j, \quad \sin 2\pi = 0$$

The following theorem contains some useful general properties of such limits.

Theorem 4.1: [Rules for Vector Limits]: If the vector functions F(t) and G(t) are functions of a real variable t and h(t) is a scalar function such that all the three functions have finite limits as $t \to t_0$, then the limit of the

- **Sum:** $\lim_{t\to t_0}[F(t)+G(t)] = \lim_{t\to t_0} F(t) + \lim_{t\to t_0} G(t)$
- **Difference:** $\lim_{t\to t_0}[F(t) - G(t)] = \lim_{t\to t_0} F(t) - \lim_{t\to t_0} G(t)$
- **Scalar multiple:** $\lim_{t\to t_0}[h(t)F(t)] = \left[\lim_{t\to t_0} h(t)\right]\left[\lim_{t\to t_0} F(t)\right]$
- **Dot product:** $\lim_{t\to t_0}[F(t).G(t)] = \left[\lim_{t\to t_0} F(t)\right].\left[\lim_{t\to t_0} G(t)\right]$
- **Cross product:** $\lim_{t\to t_0}[F(t)\times G(t)] = \left[\lim_{t\to t_0} F(t)\right]\times\left[\lim_{t\to t_0} G(t)\right]$

These limit formulas are also valid as $t \to \pm\infty$, assuming all limits exist.

ii) *Continuity of a vector function*

Definition 4.2.2: [Continuity of a Vector Function]: A vector function F(t) is said to be continuous at $t = t_0$ if

1. t_0 is in the domain of a vector function $F(t)$
2. $\lim_{t\to t_0} F(t) = F(t_0)$

Theorem 4.2: A vector function $F(t)=\left(f_1(t), f_2(t), f_3(t)\right)$ is continuous at $t=t_0$, if and only if

1. $F(t)=\left(f_1(t), f_2(t), f_3(t)\right)$ is defined at $t=t_0$.
2. all the component functions $f_1(t), f_2(t), f_3(t)$ are continuous at $t=t_0$:

   $\lim_{t\to t_0} f_1(t)=f_1(t_0)$, $\lim_{t\to t_0} f_2(t)=f_2(t_0)$, $\lim_{t\to t_0} f_3(t)=f_3(t_0)$

Example: 4.2.2:[Continuity of a Vector Function]: For what values of t is the vector function $F(t)=\left(\sin t,(1-t)^{-1}\right)$ continuous?

Solution: The components of a vector function are:

$$f_1(t)=\sin t,\quad f_2(t)=(1-t)^{-1},\quad t\in R$$

The function $f_1(t)$ is continuous for all t; $f_2(t)$ is continuous where $1-t\neq 0$ $(t\neq 1)$. Thus, F(t) is continuous, when t is a positive number other than 1. That is $t>0$, and $t\neq 1$.

Example: 4.2.3:[Continuity of a Vector Function]: For what values of t is $F(t)=\left(\sin t,(1-t)^{-1},\ln t\right)$ continuous?

Solution: The components of a vector function are:

$$f_1(t)=\sin t,\quad f_2(t)=(1-t)^{-1},\quad f_3(t)=\ln t,\quad t\in R$$

The function $f_1(t)$ is continuous for all t; $f_2(t)$ is continuous where $1-t\neq 0$ (that is, where $t\neq 1$); $f_3(t)$ is continuous for $t>0$. Thus, F (t) is continuous function whenever t is any positive number other than 1. That is $t>0, t\neq 1$.

Exercise 4.1

1. Find the domain for the following vector functions:

   a. $F(t)=2ti-3tj+t^{-1}k$

   b. $F(t)=(1-t)i+\sqrt{t}\,j-(t-2)^{-1}k$

   c. $F(t=\sin t\, i+\cos t\, j+\tan t\, k$

   d. $F(t)=\cos t\, i-\cot t\, j+\operatorname{cosec} t\, k$

   e. $F(t)+G(t)$, where $F(t)=3tj+t^{-1}k$, $G(t)=5ti+\sqrt{10-t}\,j$

   f. $F(t)-G(t)$, where $F(t)=\ln t\, i+3tj-t^2k$, $G(t)=i+5tj-t^2k$

   g. $F(t)\times G(t)$, where $F(t)=t^2i-tj+2tk$, $G(t)=(t+2)^{-1}i+(t+4)j-\sqrt{-t}k$

2. Sketch the following vector functions:

a. $F(t)=2t\,i+t^2\,j$ b. $G(t)=\sin t\,i-\cos t\,j$

3. Perform the operations of the following expressions with

$$F(t)=2t\,i-5j+t^2k,\ G(t)=(1-t)\,i+\frac{1}{t}k,\ H(t)=\sin t\,i+e^t\,j:$$

a. $2F(t)-3G(t)$ b. $F(t).G(t)$ c. $G(t).H(t)$

d. $F(t)\times H(t)$ e. $2e^tF(t)+tG(t)+10H(t)$

4. Evaluate the limits of the following expressions:

a. $\lim\limits_{t\to 1}\left[3ti+e^{2t}\,j+\sin\pi t\,k\right]$ b. $\lim\limits_{t\to 0}\left[\dfrac{\sin t\,i-tk}{t^2+t-1}\right]$

c. $\lim\limits_{t\to 1}\left[\dfrac{t^3-1}{t-1}i+\dfrac{t^2-3t+2}{t^2+t-2}j+(t^2+1)e^{t-1}k\right]$

d. $\lim\limits_{t\to 0}\left[\dfrac{te^t}{1-e^t}i+\dfrac{e^{t-1}}{\cos t}j\right]$ e. $\lim\limits_{t\to 0}\left[\dfrac{\sin t}{t}i+\dfrac{1-\cos t}{t}j+e^{1-t}k\right]$

f. $\lim\limits_{t\to 0}\left[\dfrac{\sin 3t}{\sin 2t}i+\dfrac{\ln(\sin t)}{\ln(\tan t)}j+t^tk\right]$

g. $\lim\limits_{t\to 1}\left[2ti-3j+e^tk\right]$ h. $\lim\limits_{t\to 2}\left[(2i-tj+e^tk)\times(t^2i+4\sin tj)\right]$

5. Test the continuity of the following expressions for all values of t:

a. $F(t)=ti+3j-(1-t)k$ b. $G(t)=ti-t^{-1}k$

c. $G(t)=\dfrac{i+2j}{t^2+t}$ d. $F(t)=e^t\sin t\,i+e^t\cos t\,k$

e. $F(t)=e^t\left(ti+t^{-1}j+3k\right)$ f. $G(t)=\dfrac{ti+\sqrt{t}\,j}{\sqrt{t^2+t}}$

4.3 Derivative of Vector Function

In unit-3, we defined the derivative of the scalar function f(x) which is the limit as $\Delta x\to 0$ of the difference quotient $\dfrac{\Delta f}{\Delta x}$. It was the scalar derivative of a scalar function. Exactly, the derivative of a vector functions is the limit as

$\Delta t \to 0$ of the difference quotient $\dfrac{\Delta F}{\Delta t} = \dfrac{F(t+\Delta t) - F(t)}{\Delta t}$ of a vector function F(t). This is shown in figure (4.5(a) + (b)).

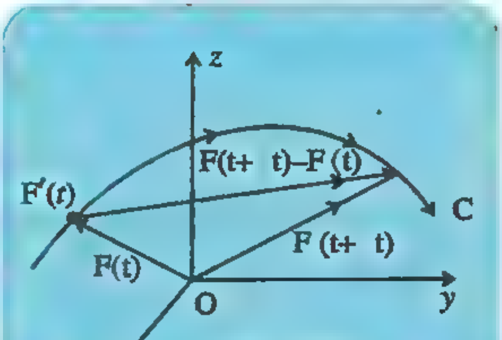


Fig. 4.5(a)

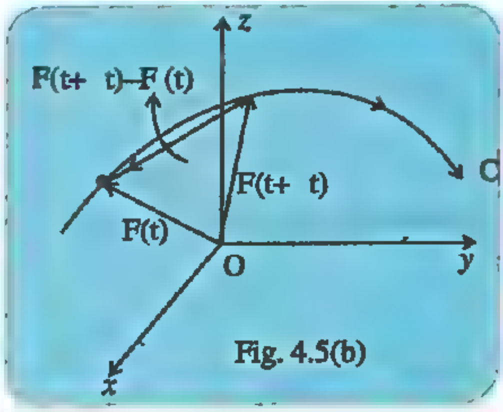


Fig. 4.5(b)

In other words, as $\Delta t \to 0$, the difference quotient $\dfrac{\Delta F}{\Delta t}$ approaches a vector $F'(t)$ which is tangent to the curve C at the terminal point of a vector F(t). This is shown in figure (4.5(c)).

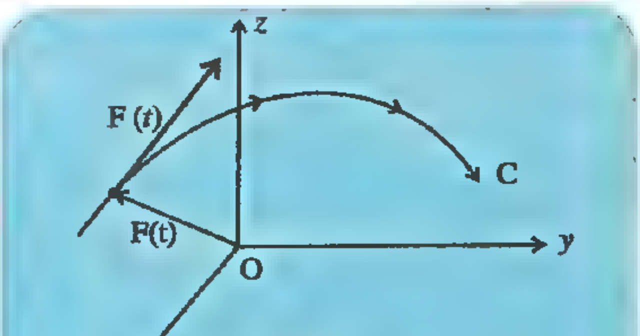


Fig. 4.5(c)

Definition 4.3.1: [Derivative of a Vector Function]: The derivative of a vector function F(t) is the vector function $F'(t)$ determined by taking the limit of a difference quotient $\dfrac{\Delta F}{\Delta t}$:

$$F'(t) = \lim_{\Delta t \to 0} \frac{\Delta F}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}, \text{ when this limit exists} \qquad (6)$$

In the Leibniz notation, the derivative of F (t) is denoted by:

$$\frac{dF}{dt} = \lim_{\Delta t \to 0} \frac{\Delta F}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t} \qquad (7)$$

In this situation we say that the vector function F(t) is differentiable at a point $t = t_o$ if $F'(t)$ is defined at $t = t_o$.

The following theorem establishes a convenient method for computing the derivative of a vector function.

Theorem 4.3: [Derivative of a Vector Function]: The vector function $F(t) = (f_1(t), f_2(t), f_3(t)) = f_1(t)i + f_2(t)j + f_3(t)k$ is differentiable at a point $t = t_o$ whenever the component functions $f_1(t), f_2(t), f_3(t)$ of F(t) are all differentiable at a point $t = t_o$:

$$F'(t) = (f_1'(t), f_2'(t), f_3'(t))$$
$$= f_1'(t)i + f_2'(t)j + f_3'(t)k$$

Proof: If a vector function F(t) is differentiable, then their component functions $f_1(t), f_2(t)$ and $f_3(t)$ exist, then the scalar derivatives $f_1'(t), f_2'(t)$ and $f_3'(t)$ by first-principle rule are:

$$F'(t) = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{[f_1(t+\Delta t)i + f_2(t+\Delta t)j + f_3(t+\Delta t)k] - [f_1(t)i + f_2(t)j + f_3(t)k]}{\Delta t}$$
$$= \left[\lim_{\Delta t \to 0} \frac{f_1(t+\Delta t) - f_1(t)}{\Delta t}\right] i + \left[\lim_{\Delta t \to 0} \frac{f_2(t+\Delta t) - f_2(t)}{\Delta t}\right] j$$
$$+ \left[\lim_{\Delta t \to 0} \frac{f_3(t+\Delta t) - f_3(t)}{\Delta t}\right] k$$
$$= f_1'(t)i + f_2'(t)j + f_3'(t)k$$

In the Leibniz notation, the derivative of F(t) is denoted by:

$$\frac{dF}{dt} = \frac{df_1}{dt} i + \frac{df_2}{dt} j + \frac{df_3}{dt} k \qquad (8)$$

Example 4.3.1: [Differentiability of a Vector Function]: For what values of t is $G(t) = |t|\, i + (\cos t)j + (t-5)k$ differentiable?

Solution: The component functions $f_2(t) = \cos t$ and $f_3(t) = t-5$ are differentiable for all values of t, but $f_1(t) = |t|$ is not (not continuous at t=0) differentiable at t=0. Thus, the vector function G(t) is differentiable for all $t \neq 0$.

Example 4.3.2: [Derivative of a Vector Function]: Find the derivative of the vector function $F(t) = e^t i + \sin t\, j + (t^3 + 5t)k$.

Solution; The derivative of a vector function with respect to t is:

$$\frac{dF}{dt} = \frac{d}{dt}(e^t)i + \frac{d}{dt}(\sin t)j + \frac{d}{dt}(t^3 + 5t)k$$

$$= e^t i + \cos t\, j + (3t^2 + 5)k$$

4.4 Vector Differentiation

Several rules for computing derivatives of vector functions are listed below, which can be proved by applying rules for limits of vector functions to appropriate theorems for scalar derivatives.

i) *Proof of Vector Differentiation Formulae*

If $F(t) = f_1(t)i + f_2(t)j + f_3(t)k$

$$= \left(f_1(t), f_2(t), f_3(t)\right)$$

and $G(t) = g_1(t)i + g_2(t)j + g_3(t)k$

$$= \left(g_1(t), g_2(t), g_3(t)\right)$$

are the differentiable vector functions, h(t) is a differentiable scalar function, and a and b are any scalar quantities, then the differential rule is:

- **Constant rule:** $\frac{d}{dt}(w) = \frac{d}{dt}(w_1 i + w_2 j + w_3 k) = 0,$

  w is a constant vector function

- **Linearity rule:** $\frac{d}{dt}(aF \pm bG)(t) = a\frac{dF}{dt} \pm b\frac{dG}{dt}$

Proof:

$$\frac{d}{dt}\left[(aF \pm bG)(t)\right] = \frac{d}{dt}\left[\left(af_1(t), af_2(t), af_3(t)\right) \pm \left(bg_1(t), bg_2(t), bg_3(t)\right)\right]$$

$$= \frac{d}{dt}\left[af_1(t) \pm bg_1(t), af_2(t) \pm bg_2(t), af_3(t) + bg_3(t)\right]$$

$$= \left[af_1'(t) \pm bg_1'(t), af_2'(t) \pm bg_2'(t), af_3'(t) \pm bg_3'(t)\right]$$

$$= a\left(f_1'(t), f_2'(t), f_3'(t)\right) \pm b\left(g_1'(t), g_2'(t), g_3'(t)\right)$$

$$= aF'(t) \pm bG'(t)$$

- **Scalar Multiple rule:** $\frac{d}{dt}\left((hF)(t)\right) = \frac{dh}{dt}F + h\frac{dF}{dt}$

Proof:

$$\frac{d}{dt}(hF)(t) = \frac{d}{dt}\left[h(t)f_1(t), h(t)f_2(t), h(t)f_3(t)\right]$$

$$= h'(t)f_1(t) + h(t)f_1'(t) + h'(t)f_2(t) + h(t)f_2'(t) + h'(t)f_3(t) + h(t)f_3'(t))$$

$$= [h'(t)f_1(t) + h'(t)f_2(t) + h'(t)f_3(t)] + [h(t)f_1'(t) + h(t)f_2'(t) + h(t)f_3'(t)]$$

$$= \frac{dh}{dt}F + h\frac{dF}{dt}$$

- **Quotient rule:** $\frac{d}{dt}\left(\frac{f}{h}\right)(t) = \frac{1}{h^2}\left[h\frac{dF}{dt} - \frac{dh}{dt}F\right]$

- **Dot Product rule:** $\frac{d}{dt}\left((F.G)(t)\right) = \frac{dF}{dt}.G + F.\frac{dG}{dt}$

Proof:

$$\frac{d}{dt}(F.G)(t) = \frac{d}{dt}\left[f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t)\right]$$

$$= f_1'(t)g_1(t) + f_1(t)g_1'(t) + f_2'(t)g_2(t) + f_2(t)g_2'(t) + f_3'(t)g_3(t) + f_3(t)g_3'(t)$$

$$= [f_1'(t)g_1(t) + f_2'(t)g_2(t) + f_3'(t)g_3(t)] + [f_1(t)g_1'(t) + f_2(t)g_2'(t) + f_3(t)g_3'(t)]$$

$$= \frac{dF}{dt}\cdot G + F\cdot\frac{dG}{dt}$$

- **Cross Product rule:** $\frac{d}{dt}(F \times G)(t) = \frac{dF}{dt} \times G + F \times \frac{dG}{dt}$

- **Chain rule:** $\frac{d}{dt}\left(F(h(t))\right) = \frac{dF(h(t))}{dt}\frac{dh(t)}{dt}$

Proof:

$$\frac{d}{dt}(F(h(t))=\frac{d}{dt}[f_1(h(t)), f_2(h(t)), f_3(h(t))]$$
$$= f_1'(h(t))h'(t)+ f_2'(h(t))h'(t)+ f_3'(h(t))h'(t)$$
$$=[f_1'(h(t)), f_2'(h(t)), f_3'(h(t))]h'(t)]$$
$$=\frac{dF(h(t))}{dt}\left(\frac{dh(t)}{dt}\right)$$

We leave the proofs of quotient and cross product rules as exercises.

Example 4.4.1:[Derivative of a Cross Product]: Let $F(t)= i+tj+t^2k$ and $G(t)= ti+e^t j+3k$ are the vector functions. Verify the derivative:

$$\frac{d}{dt}(F\times G)(t)=\frac{dF}{dt}\times G+F\times\frac{dG}{dt}$$

Solution: For verification, the LHS is:

$$LHS=\frac{d}{dt}(F\times G)(t)$$
$$=\frac{d}{dt}\begin{vmatrix} i & j & k \\ 1 & t & t^2 \\ t & e^t & 3 \end{vmatrix}$$
$$=\frac{d}{dt}[(3t-t^2e^t)i-(3-t^3)j+(e^t-t^2)k]$$
$$=(3-2te^t-t^2e^t)i+3t^2j+(e^t-2t)k$$

The expressions

$$\frac{dF}{dt}=j+2tk,\quad \frac{dG}{dt}=i+e^tj$$
$$\frac{dF}{dt}\times G=\begin{vmatrix} i & j & k \\ 0 & 1 & 2t \\ t & e^t & 3 \end{vmatrix}=(3-2te^t)i-(-2t^2)j+(-t)k$$

$$F\times\frac{dG}{dt}=\begin{vmatrix} i & j & k \\ 1 & t & t^2 \\ 1 & e^t & 0 \end{vmatrix}=(-t^2e^t)i-(-t^2)j+(e^t-t)k$$

are used in the RHS to obtain

$$RHS=\frac{dF}{dt}\times G+F\times\frac{dG}{dt}$$
$$=(3-2te^t-t^2e^t)i+(2t^2+t^2)j+(-t+e^t-t)k$$
$$=(3-2te^t-t^2e^t)i+(3t^2)j+(e^t-2t)k$$

which is identical to the LHS. Thus, the LHS=RHS.

Example 4.4.2:[Derivative of Vector Functions]: If $F(t) = i + e^t j + t^2 k$ and $G(t) = 3t^2 i + e^{-t} j - 2t k$ are the two vector functions and h(t) is any scalar function, then evaluate the following derivatives:

a. $\frac{d}{dt}\left(2F + t^3 G\right)$ b. $\frac{d}{dt}(F.G)$

Solution:

a.
$$\frac{d}{dt}\left(2F + t^3 G\right) = \frac{d}{dt}\left[2\left(i + e^t j + t^2 k\right) + t^3\left(3t^2 i + e^{-t} j - 2tk\right)\right]$$
$$= \frac{d}{dt}\left[(2 + 3t^5)i + (2e^t + t^3 e^{-t})j + (2t^2 - 2t^4)k\right]$$
$$= 15t^4 i + \left(2e^t + 3t^2 e^{-t} - t^3 e^{-t}\right) j + (4t - 8t^3)k$$
$$= 15t^4 i + \left(2e^t + t^2 e^{-t}(3 - t)\right) j + 4t(1 - 2t^2)k$$

b.
$$\frac{d}{dt}(F.G) = \frac{d}{dt}\left[(i + e^t j + t^2 k).(3t^2 i + e^{-t} j - 2tk)\right],\ i.i = j.j = k.k = 1$$
$$= \frac{d}{dt}(3t^2 + 1 - 2t^3) = 6t + 0 - 6t^2 = -(6t^2 - 6t)$$

ii) *Position vector, velocity and acceleration*

Definition 4.4.1: An object that moves in such a way that its position at any time t is said to have a

1. **position vector** or displacement R(t)
2. **velocity** $V = \frac{dR}{dt} = R'(t)$
3. **acceleration** $A = \frac{dV}{dt} = \frac{d^2R}{dt^2} = R''(t)$

At any time t, **the speed** is $|V| = |v_1 i + v_2 j + v_3 k| = \sqrt{v_1^2 + V_2^2 + V_3^2}$, the magnitude of the velocity and **the direction of motion** is $\frac{V}{|V|}$.

Example 4.4.3: [Speed and Direction of a Particle]: A particle's position at time t is determined by the vector $R(t) = \cos ti + \sin tj + t^3 k$. Find the particle's velocity, speed, direction and acceleration at a time t=2. Interpret the particle's motion.

Solution: If the particle's position at a time t is , then $R(t) = \cos t\, i + \sin t\, j + t^3 k$ then, the particle's velocity and acceleration are:

$$V(t)=\frac{dR}{dt}=\frac{d}{dt}\left[\cos t\, i+\sin t\, j+t^3k\right]=-\sin t\, i+\cos t\, j+3t^2k$$

$$A(t)=\frac{dV}{dt}=\frac{d}{dt}\left(\frac{dR}{dt}\right)$$

$$=\frac{d}{dt}\left[-\sin t\, i+\cos t\, j+3t^2k\right]=-\cos t\, i-\sin t\, j+6tk$$

The velocity at a time t=2 is

$$V(2)=-\sin 2i+\cos 2j+3(4)k\approx-0.91i-0.42j+12k,\ \text{use radians}$$

The acceleration at a time t=2 is

$$A(2)=-\cos 2i-\sin 2j+6(2)k\approx 0.42i-0.91j+12k$$

The speed is $|V|=\sqrt{(-\sin t)^2+(\cos t)^2+(3t^2)^2}=\sqrt{1+9t^4}$ .At a time t=2, the speed is $|V|=\sqrt{2+9(2)^4}=\sqrt{145}$.

The direction of motion is:

$$\frac{V}{|V|}=\frac{1}{\sqrt{145}}\left[-\sin t\, i+\cos t\, j+3t^2k\right]$$

At a time t=2, the direction of motion is:

$$\frac{V}{|V|}=\frac{1}{\sqrt{145}}\left[-\sin 2i+\cos 2j+12k\right]\approx-0.91i-0.42j+12k$$

Exercise 4.2

1. Find the vector derivative $F'(t)$ of the following vector functions:

a. $F(t)=ti+t^2j+(t+t^3)k$ b. $F(s)=(si+s^2j+s^2k)+(2s^2i-sj+3k)$

c. $F(\theta)=\cos\theta\left[i+\tan\theta j+3k\right]$

2. Find $F'(t)$ and $F''(t)$ of the following vector functions:

a. $F(t)=t^2i+t^{-1}j+e^{2t}k$ b. $F(s)=(1-2s^2)i+s(\cos s)j-sk$

c. $F(s)=\sin si+\cos sj+s^2k$ d. $F(\theta)=\sin^2\theta i+\cos 2\theta j+\theta^2k$

3. Differentiate the following scalar functions:

a. $f(x)=[xi+(x+1)j].[2xi-3x^2j]$

b. $f(x)=[\cos xi+xj-xk].[\sec xi-x^2j+2xk]$

c. $g(x)=|\sin xi-2xj+\cos xk|$

4. Find the particle's velocity, acceleration, speed and direction of motion for the indicated value of t, when the position vector of a particle in space at time t is R(t):

a. $R(t)=ti+t^2j+2tk\ \ at\ t=1$

b. $R(t)=(1-2t)i-t^2j+e^tk\ \ at\ t=0$

c. $R(t)=\cos t\,i+\sin t\,j+3tk\ \ at\ t=\pi/4$

5. If $v=2i-j+5k$ and $w=i+2j-3k$ are the two vector functions, then evaluate the following derivatives problems:

a. $\frac{d}{dt}(v+tw)$

b. $\frac{d^2}{dt^2}(v\,.\,t^4w)$

c. $\frac{d^2}{dt^2}(t|v|+t^2|w|)$

d. $\frac{d}{dt}(tv\times t^2w)$

6. Verify the following indicated equations for the vector functions, when

$F(t)=(3+t^2)i-(\cos 3t)j+t^{-1}k$ and $G(t)=\sin(2-t)i-e^{2t}k$:

a. $(3F-2G)'(t)=3F'(t)-2G'(t)$

b. $(F.G)'(t)=(F'.G)(t)+(F.G')(t)$

7. If F(t) and G(t) are differentiable vector functions of t, then prove that

a. $(F.G)'(t)=(F'.G)(t)+(F.G')(t)$

b. $(F\times G)'(t)=(F'\times G)(t)+(F\times G')(t)$

8. If F(t) is a differentiable vector functions of t such that $F(t)\neq 0$, then show that

$$\frac{d}{dt}\frac{F(t)}{|F(t)|}=\frac{F'(t)}{|F(t)|}-\frac{[F(t).F'(t)]F(t)}{|F(t)|^3}$$

Glossary

- The **parametric equation** for the plane curve C generated by the set of ordered pairs in 2-space is:
  $(x, y)=(x(t), y(t)) = (f(t), g(t))$
- The **parametric equation** for the plane curve C generated by the set of ordered triples in 3-space is:
  $(x, y, z) = (x(t), y(t), z(t)) = (f(t), g(t), h(t))$
- **Continuity of a Vector Function:** A vector function F(t) is **continuous** at $t = t_0$ if
  t_0 is in the domain of F(t)
  $\lim_{t \to t_0} F(t) = F(t_0)$
- **Derivative of a Vector Function:** The derivative of a vector function $F(t)$ is the vector function $F'(t)$ determined by the limit
  $$F'(t)= \lim_{\Delta t \to 0} \frac{\Delta F}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t)-F(t)}{\Delta t},$$
  whenever this limit exists. In the Leibniz notation, the derivative of F(t) is denoted by:
  $$\frac{dF}{dt} = \lim_{\Delta t \to 0} \frac{\Delta F}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t)-F(t)}{\Delta t}$$
- If an object moves in such a way that its position at any time t is the **position vector** or displacement R(t), then the
  - velocity is $V = \frac{dR}{dt} = R'(t)$
  - acceleration is $A = \frac{dV}{dt} = \frac{d^2R}{dt^2} = R''(t)$
  - At any time t, the speed is
    $|V| = |V_1 i + v_2 j + v_3 k| = \sqrt{v_1^2 + V_2^2 + V_3^2}$, the magnitude of the velocity and the **direction of motion** is $\frac{V}{|V|}$.

Unit

5 INTEGRATION

This unit tlls us, how to:

- demonstrate the concept of the integral as an accumulator.
- know integration as the inverse process of differentiation antiderivative); explain the constant of integration; know standard integrals from standard differentiation formulae.
- recognize the different rules of integration.
- use standard differentiation formulae for the proof of the given standard integrals.
- explain the method of integration by substitution, and its use to evaluate indefinite integrals.
- apply method of substitution for evaluating integrals of some different types.
- recognize integration by parts formula and its use to evaluate integrals of some different types.
- evaluate integrals using integration by parts.
- use partial fractions in integration and its use in different types of indefinite integrals.
- define the definite integral as the limit of a sum.
- describe the fundamental theorem of calculus and its recognization in different situations.
- represent the definite integral as the area under a curve and its use in different situations.
- apply definite integrals to calculate the area under a curve.

5.1 Introduction

In the previous units 4 and 5, we studied the derivative of a function and various applications of derivatives. That material belongs to the branch of calculus is called **differential calculus.** In this unit, we will study another branch of calculus, which is called **integral calculus.** Like the derivative of a function, the definite integral of a function is a special limit with many diverse applications. Geometrically, the derivative is related to the slope of the tangent line to the curve, while the reverse of derivative is the definite integral which is related to the area under a curve.

i) *The accumulator concept of integration*

Functions used in applications in previous units have provided information about a total amount of a quantity, such as cost, revenue, profit, temperature,

gallons of oil or distance. Derivatives of these functions provided information about the rate of change of these quantities and allowed us to answer important questions about the extrema of the functions. **It is not always possible to find ready-made functions that provide information about the total amount of a quantity**, but it is often possible to collect enough data to come up with a function that gives the rate of change of a quantity.

Accumulative point of view is that the derivative gives the rate of change when the total amount is known; the reverse process of derivative gives the total amount of a quantity, when the rate of change of a quantity is known. This reverse process of derivative or **antiderivative** or **antidifferentiation** (inverse process of differentiation) is the main topic of this unit.

ii) Integration as inverse process of differentiation

If F(x) is any unknown function and its derivative is $F'(x)=f(x)$, say, then the reverse process is to give a function F(x) such that its derivative $F'(x)$ is equal to f(x):

$$F'(x)=f(x) \qquad (1)$$

Definition 5.1.1: [Antiderivative]: A function F(x) is called an antiderivative of a function f(x) on the interval [a,b], if at all points of the interval, the identity $F'(x)=f(x)$ is true.

Example 5.1.1:[Antiderivative]: Find the antiderivative of a function $f(x)=x^2$.

Solution: From the definition of an antiderivative, it follows that the function $F(x)=\frac{x^3}{3}$ has an antiderivative of $f(x)=x^2$, since $F'(x)=f(x)=x^2$.

iii) The concept of integration

It is easy to see that if the given function f(x) has an antiderivative, then this antiderivative will not be the only one. In the Example 5.1.1, we will take the following functions as antiderivatives

$$F'(x)=\frac{x^3}{3}+1,\quad F(x)=\frac{x^3}{3}-7\,\ldots\ldots,\; F(x)=\frac{x^3}{3}+C,$$

of a function $f(x)=x^2$, since $F'(x)=f(x)$.

It may be proved that the functions of the form $\frac{x^3}{3}+C$ for any constant value of C exhaust all antiderivatives of the function $f(x)=x^2$. This develops a consequence of the theorem:

Theorem 5.1: If $F_1(x)$ and $F_2(x)$ are two antiderivatives of a function f(x) on an interval [a, b], then the difference between them is a constant quantity.

Proof: If $F_1'(x)$ and $F_2'(x)$ are the derivatives of $F(x)$ and $F_2(x)$, then by virtue of the definition of an antiderivative, we have

$$F_1(x)=f(x),\ \ F_2'(x)=f(x) \qquad (2)$$

If we put the difference of $F_1(x)$ and $F_2(x)$ by

$$F_1(x)-F_2(x)=\phi(x),$$

then by virtue of the definition of an antiderivative for any value of x on the interval [a, b]:

$$F_1'(x)-F_2'(x)=f(x)-f(x)$$

$$\frac{d}{dx}\left(F_1(x)-F_2(x)\right)=0 \Rightarrow \frac{d}{dx}\left(\phi(x)\right)=0 \Rightarrow \phi'(x)=0$$

From $\phi'(x)=0$, it follows that $\phi(x)=C$ is any constant quantity. It follows that if a given function f(x) has an antiderivative F(x), then any other antiderivative of f(x) will be of the form F(x)+C, for any constant value of C.

The process of finding antiderivative is then called the **antidifferentiation or inverse of differentiation** or **integration.**

Definition 5.1.2:[Indefinite Integral]: If the function F(x) is an antiderivative of f(x), then the expression F(x)+C is of course the **indefinite integral** of the function f(x) and is denoted by the symbol:

$$\int f(x)dx=F(x)+C \quad \text{if} \quad F'(x)=f(x) \qquad (3)$$

The symbol $\int$ is called an **integral sign** and the function $f(x)$ is called the **integrand**. The symbol dx indicates that the antiderivative is performed with respect to the variable x. The arbitrary constant C is called the **constant of integration.** Referring to the preceding discussion, we can write

$$\int 2x dx=x^2+C, \qquad \text{since} \qquad \frac{d}{dx}(x^2+C)=2x$$

The variables other than x can also be used in indefinite integrals. For example,

$$\int 2t\,dt = t^2 + C \text{, since } \frac{d}{dt}(t^2 + C) = 2t,$$

$$\int 2u\,du = u^2 + C \text{, since } \frac{d}{du}(u^2 + C) = 2u$$

iv) Standard integrals through standard differentiation formulae

If f(x) is any function of x, then in different situations, the integral of f(x) can be found directly in light of definition 5.1.2 from the chart of integrals and its differentiation formulae. The truth of the integrals can easily be checked by differentiating the right side of the integral which always equals the integrand.

Integrals	Differentiation Formulae
1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = x^n,\ n \neq -1$
2. $\int \frac{dx}{x} = \ln\lvert x\rvert + C,\ n = -1$	$\frac{d}{dx}(\ln x + C) = \frac{1}{x}$
3. $\int \cos kx\,dx = \frac{\sin kx}{k} + C$	$\frac{d}{dx}\left(\frac{\sin kx}{k} + C\right) = \cos kx$
4. $\int \sin kx\,dx = \frac{-\cos kx}{k} + C$	$\frac{d}{dx}\left(\frac{-\cos kx}{k} + C\right) = \sin kx$
5. $\int \frac{dx}{\cos^2 x} = \tan x + C$	$\frac{d}{dx}(\tan x + C) = \sec^2 x = \frac{1}{\cos^2 x}$
6. $\int \frac{dx}{\sin^2 x} = -\cot x + C$	$\frac{d}{dx}(-\cot x + C) = cosec^2 x = \frac{1}{\sin^2 x}$
7. $\int \tan dx = -\ln\lvert\cos x\rvert + C$	$\frac{d}{dx}(-\ln\cos x + C) = \tan x$
8. $\int \cot x\,dx = \ln\lvert\sin\rvert + C$	$\frac{d}{dx}(\ln\sin x + C) = \cot x$

9	$\int e^{mx}\,dx = \dfrac{e^{mx}}{m} + C$	$\dfrac{d}{dx}\left(\dfrac{e^{mx}}{m} + C\right) = e^{mx}$
10	$\int a^x\,dx = \dfrac{a^x}{\ln a} + C$	$\dfrac{d}{dx}\left(\dfrac{a^x}{\ln a} + C\right) = a^x$
11	$\int \dfrac{dx}{1+x^2} = \tan^{-1} x + C$	$\dfrac{d}{dx}\left(\tan^{-1} x + C\right) = \dfrac{1}{1+x^2}$
12	$\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C$	$\dfrac{d}{dx}\left(\dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C\right) = \dfrac{1}{a^2+x^2}$
13	$\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\ln\left\|\dfrac{a+x}{a-x}\right\| + C$	$\dfrac{d}{dx}\left(\dfrac{1}{2a}\ln\left\|\dfrac{a+x}{a-x}\right\| + C\right) = \dfrac{1}{a^2-x^2}$
14	$\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\ln\left\|\dfrac{x-a}{x+a}\right\| + C$	$\dfrac{d}{dx}\left(\dfrac{1}{2a}\ln\left\|\dfrac{x-a}{x+a}\right\| + C\right) = \dfrac{1}{x^2-a^2}$
15.	$\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$	$\dfrac{d}{dx}\left(\sin^{-1} x + C\right) = \dfrac{1}{\sqrt{1-x^2}}$
16.	$\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\dfrac{x}{a} + C$	$\dfrac{d}{dx}\left(\sin^{-1}\dfrac{x}{a} + C\right) = \dfrac{1}{\sqrt{a^2-x^2}}$
17	$\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left\|x + \sqrt{x^2 \pm a^2}\right\| + C$	$\dfrac{d}{dx}\left(\ln\left\|x + \sqrt{x^2 \pm a^2}\right\| + C\right) = \dfrac{1}{\sqrt{x^2 \pm a^2}}$

Example 5.1.2: [Formulae Chart]: Use the chart of integral folmulae to evaluate the following integrals:

a. $\int x^{1/3} dx$ **b.** $\int 1/x^3 dx$

Solution:

a. Formula serial number (1) for n=1/2 is used to obtain:

$$\int x^{1/3} dx = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = \frac{x^{4/3}}{4/3} + C = \frac{3}{4}x^{4/3} + C$$

b. Formula (1) for n=-3 is used to obtain:

$$\int \frac{1}{x^3} dx = \frac{x^{-3+1}}{-3+1} + C = \frac{x^{-2}}{-2} + C = \frac{-1}{2x^2} + C$$

5.2.1 Rules of Integration

i) Recognization of rules of integration-I

- ***Recognization of*** $\frac{d}{dx}\left[\int f(x)dx\right]=f(x)$

It is extremely important to recognize that differentiation is the inverse operation of integration

$$\frac{d}{dx}\left[\int f(x)dx\right]=f(x) \qquad (4)$$

and the integration is the inverse operation of differentiation:

$$\int f'(x)dx=f(x)+C \qquad (5)$$

The notations (4) and (5) are illustrated by the following examples. Let us begin with (4). If $f(x)=x^3$, then the integral of f(x) is:

$$\int x^3 dx=\frac{x^4}{4}+C$$

Now the derivative of the integral is:

$$\frac{d}{dx}\left[\frac{x^4}{4}+C\right]=\frac{4x^3}{4}=x^3$$

This illustrates that the derivative of the integral of f(x) is equal to f(x). Now look at (5). If $f(x)=x^3$ and $f'(x)=3x^2$, then the integral of $f'(x)$ is

$$\int 3x^2 dx=\frac{3x^3}{3}+C=x^3+C$$

This illustrates that the integral of the derivative of f(x) is equal to f(x) plus a constant quantity C.

This is summarized in a box:

If all integrals exist, then:

1. $\frac{d}{dx}[\int f(x)dx]=f(x)$ **2.** $\int f'(x)dx=f(x)+C$

- ***Recognization of*** $\int kf(x)\,dx = k\int f(x)dx$

A constant factor may be taken outside the integral sign. If k is any real constant, then

$$\int k\,f(x)\,dx = k\int f(x)dx \qquad (6)$$

To establish that the expressions to the right and left are the same in the sense indicated above, it is necessary to prove that their derivatives with respect to x are equal. Then the derivative of (6) w.r.t x is:

$$\frac{d}{dx}\left[\int k\,f(x)\,dx\right] = \frac{d}{dx}\left[k\int f(x)\right]$$
$$kf(x) = kf(x) \qquad (7)$$

The derivatives of the right and left sides of (7) are equal.

- ***Recognization of*** $\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

The indefinite integral of an algebraic sum of two or more functions is equal to the algebraic sum of their integrals:

$$\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx \qquad (8)$$

The derivative of (8) w.r.t x is:

$$\frac{d}{dx}\left[\int[f(x)+g(x)]\right]dx = \frac{d}{dx}\left[\int f(x)dx + \int g(x)dx\right]$$
$$f(x)+g(x) = f(x)+g(x) \qquad (9)$$

The derivatives of the right and left sides of (9) are equal, therefore as in (8).
These rules are summarized in a box:

1. Constant Rule:	$\int kdx = kx + C$, k is constant
2. Constant Multipl Rule:	$\int k\,f(x)dx = k\,f(x) + C$
3. Sum Rule:	$\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx + C$
4. Power Rule:	$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
5. Logarithmic Rule:	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C,\ x \neq 0$

Example 5.2.1:[Constant Rule]: Evaluate the following integrals:

a. $\int(10+a)dx$ b. $\int(10+a)x^2dx$ c. $\int(5x^2+4x+2)dx$ d. $\int 4x^{-8}dx$

Solution:

a. The constant rule of integration is used to obtain:

$$\int(10+a)dx=(10+a)x+C$$

b. The constant multiple rule of integration is used to obtain:

$$\int(10+a)x^2dx=(10+a)\frac{x^3}{3}+C=\frac{(10+a)x^3}{3}+C$$

c. The sum rule of integration is used to obtain:

$$\int(5x^2+4x+2)dx=\int 5x^2dx+\int 4xdx+\int 2dx+C$$

$$=5\frac{x^3}{3}+4\frac{x^2}{2}+2x+C$$

$$=\frac{5x^3}{3}+\frac{4x^2}{2}+2x+C=\frac{5x^3}{3}+2x^2+2x+C$$

d. Power rule of integration is used to obtain:

$$\int 4x^{-8}dx=4\int x^{-8}dx=4\frac{x^{-8+1}}{-8+1}+C=\frac{4x^{-7}}{-7}+C=-\frac{4x^{-7}}{7}+C$$

ii) *Recognization of rules of integration-II*

- ***Recognization of*** $\int[f(x)]^n f'(x)dx=\frac{(f(x))^{n+1}}{n+1}+C,\ n\neq -1$

Let it will be requied to find the integral

$$\int[f(x)]^n dx=? \quad (10)$$

For this integral, we are not in position to directly select the antiderivative of f(x), however the integral exists. In this situation, we need to change the variable x in the expression under the integral sign by putting:

x=u(t) and $\frac{dx}{dt}=u'(t)$

These are used in (10) to obtain:

$$\int[f(x)]^n dx=\int[f[u(t)]]^n u'(t)dt \quad (11)$$

To establish that the expressions to the right and left are the same in the sense indicated above, it is necessary to prove that their derivatives with respect to x are equal. Then the derivative of (11) w.r.t x is:

$$\frac{d}{dx}\left[\int [f(x)]^n\, dx\right] = \frac{d}{dx}\left[\int [f[u(t)]]^n u'(t)dt\right]$$

$$[f(x)]^n = \frac{d}{dx}\left[\int [f[u(t)]]^n u'(t)dt\right] \times \frac{dt}{dt}$$

$$= \frac{d}{dt}\left[\int [f[u(t)]]^n u'(t)dt\right] \times \frac{dt}{dx}$$

$$= [f[u(t)]]^n u'(t) \times \frac{1}{u'(t)}, \quad dx/dt = u'(t)$$

$$= [f[u(t)]]^n = [f(x)]^n$$

The derivatives of the right and left sides of the expression are equal, therefore, as in (11).

- ***Recognization of*** $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$

Let it will be requied to find the integral

$$\int \frac{1}{f(x)} dx = ? \qquad (12)$$

We need to change the variable x in the expression under the integral sign by putting:

x=u(t) and $\frac{dx}{dt} = u'(t)$

These are used in (12) to obtain:

$$\int \frac{1}{f(x)} dx = \int \left[\frac{1}{f[u(t)]}\right] u'(t)dt \qquad (13)$$

To establish that the expressions to the right and left are the same in the sense indicated above, it is necessary to prove that their derivatives with respect to x are equal. Then the derivative of (12) w.r.t x is:

$$\frac{d}{dx}\left[\int \frac{1}{f(x)}dx\right]=\frac{d}{dx}\left[\int\left[\frac{1}{f[u(t)]}\right]u'(t)dt\right]$$

$$=\frac{d}{dx}\left[\int\left[\frac{1}{f[u(t)]}\right]u'(t)dt\right]\times\frac{dt}{dt}$$

$$=\frac{d}{dt}\left[\int\left[\frac{1}{f[u(t)]}\right]u'(t)dt\right]\times\frac{dt}{dx}$$

$$=\left[\frac{1}{f[u(t)]}\right]u'(t)\times\frac{1}{u'(t)}$$

$$=\frac{1}{f[u(t)]}=\frac{1}{f(x)}$$

The derivatives of the right and left sides of the expression are equal, therefore, as in (13).

- ***Recognization of*** $\int e^{f(x)} f'(x)dx$

Let it will be requied to find the integral

$$\int e^{f(x)}dx=? \qquad (14)$$

We need to change the variable x in the expression under the integral sign by putting:

x=u(t) and $\frac{dx}{dt}=u'(t)$

These are used in (14) to obtain:

$$\int e^{f(x)}dx=\int e^{f[u(t)]}u'(t)dt \qquad (15)$$

To establish that the expressions to the right and left are the same in the sense indicated above, it is necessary to prove that their derivatives with respect to x are equal. Then the derivative of (14) w.r.t x is:

$$\frac{d}{dx}\left[\int e^{f(x)}dx\right]=\frac{d}{dx}\left[\int e^{f[u(t)]}u'(t)dt\right]$$

$$e^{f(x)}=\frac{d}{dx}\left[\int e^{f[u(t)]}u'(t)dt\right]\times\frac{dt}{dt}$$

$$=\frac{d}{dt}\left[\int e^{f[u(t)]}u'(t)dt\right]\times\frac{dt}{dx}$$

$$=e^{f[u(t)]}u'(t)\times\frac{1}{u'(t)}=e^{f[u(t)]}=e^{f(x)}$$

The derivatives of the right and left sides of the expression are equal, therefore, as in (15).

5.3 Integration by Substitution

i) *Method of integration by substitution*

In previous section, we saw how to integrate a few simple functions. More complicated functions can sometimes be integrated by substitution. The technique depends on the idea of a differential. If $u = f(x)$, then, the differential of u, written du, is defined as

$$du = f'(x)dx.$$

Differentials have many useful interpretations which are studied in more advanced courses. We shall only use them as a convenient notational device when finding an antiderivative such as

$$\int x^2\sqrt{x^3+1}dx \tag{16}$$

The function $x^2\sqrt{x^3+1}$ is reminiscent of the chain rule and so we shall try to use differentials and the chain rule in reverse to find the antiderivative. Let $u = x^3+1$, then $du = 3x^2dx$. Now substitute u for x^3+1 and du for $3x^2dx$ in the indefinite integral (16) to obtain

$$\int x^2\sqrt{x^3+1}dx=\int\frac{\sqrt{u}}{3}du, \quad u=x^3+1,\ du/dx=3x^2$$

$$=\frac{1}{3}\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}+C$$

$$= \frac{1}{3}\left(\frac{2}{3}\right)u^{\frac{3}{2}} + C = \frac{2}{9}u^{\frac{3}{2}} + C = \frac{2}{9}\left(x^3+1\right)^{\frac{3}{2}} + C$$

This method of integration is called **integration by substitution**. As shown above, it is simply the chain rule for derivative in reverse. The results can always be verified by differentiation.

General Indefinite Integral Formulae:

If u is a function of x, where u=f(x) and $du = f'(x)dx$, then

1. $\int [f(x)]^n f'(x)dx = \int [u(x)]^n du = \frac{[u(x)]^{n+1}}{n+1} + C,\ n \neq -1$

2. $\int \frac{f'(x)}{f(x)}dx = \int \frac{du}{u(x)} = \ln|u(x)| + C$

3. $\int e^{f(x)} f'(x)dx = \int e^{u(x)}du = e^{u(x)} + C$

If k is a real number, $k \neq 0$, then

1. $\int e^x dx = e^x + C$ 2. $\int e^{kx}dx = \frac{e^{kx}}{k} + C$ 3. $\int a^x dx = \frac{a^x}{\ln a} + C$

ii) *Method of substitution to evaluate the indefinite integrals*

Example:5.3.1:[Product/Quotient/Exponential]: Evaluate the following integrals:

a. $\int \left(x^3 - 5x + 7\right)^4 \left(3x^2 - 5\right)dx$ b. $\int \frac{dx}{9x+7}$ c. $\int e^{(3x+2)}dx$

Solution:

a. We need to substitute a new variable u(x) :

$x^3 - 5x + 7 = u$

$$\frac{d}{dx}(x^3 - 5x + 7) = \frac{du}{dx} \Rightarrow 3x^2 - 5 = \frac{du}{dx} \Rightarrow (3x^2 - 5)dx = du \quad (17)$$

Substitute (17) in the given integral to obtain:

$$\int (x^3-5x+7)^4(3x^2-5)dx = \int u^4 du$$

$$= \frac{(u)^5}{5} + C$$

$$= \frac{1}{5}(x^3-5x+7)^5 + C \quad u = x^3-5x+7$$

b. We need to substitute a new variable $u(x)$:

$9x+7=u$

$$\frac{d}{dx}(9x+7) = \frac{du}{dx} \Rightarrow 9 = \frac{du}{dx} \Rightarrow dx = \frac{du}{9} \qquad (18)$$

Substitute (18) in the given integral to obtain:

$$\int \frac{dx}{9x+7} = \int \frac{du}{9u} = \frac{1}{9}\ln u + C = \frac{1}{9}\ln(9x+7) + C. \quad u = 9x+7$$

c. We need to substitute a new variable u(x) :

$$3x+2=u,\ \frac{d}{dx}(3x+2) = \frac{du}{dx} \Rightarrow 3 = \frac{du}{dx} \Rightarrow dx = \frac{du}{3} \qquad (19)$$

Substitute (19) in the given integral to obtain:

$$\int e^{(3x+2)}dx = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{(3x+2)} + C. \quad u = 3x+2$$

Example 5.3.2:[Cost of an Item]: An item is purchased at a rate of Rs.2. Find the total cost $y(x)$ of the x number of items.

a. Determine the fixed cost (the constant of integration, that is C), when the cost of $x=4$ number of items is rupees $y=20$.

b. Determine the specific cost for $x=10$ number of items.

Solution: The rate at which x number of items purchased is $y'(x)=2$. The total cost $y(x)$ of x number of items is the indefinite integral of $y(x)$ with respect to x:

$$\int y'(x)dx = \int 2dx$$

$$y(x) = 2x + C, \quad C \text{ is the fixed cost} \qquad (20)$$

In geeral, equation (20) is the (general curve) cost of x number of items. The fixed cost C is obtained by inserting x=4 and y=20 in equation (20)

$$y(x) = 2x + C \Rightarrow 20 = 2(4) + C \Rightarrow C = 12$$

a. Use this value of C=12 (fixed cost means transportation charges etc) in equation (20) to obtain the specific (particular curve) cost of x number of items

$$y(x)=2x+12 \qquad (21)$$

b. The specific cost of x=10 numbers of items is obtained by inserting x=10 in equation (21):

$y(10)=2(10)+12=32$ Rupees

Example 5.3.3:[Wound Healing]: If the area A of a healing wound changes at a rate approximateld by,

$$\frac{dA}{dt}=-4t^{-3},\quad 1\le t\le 10,$$

t is time in days and the wound area is A=2 square centimeters on day t=1. What will the area of the wound be in 10 days?

Solution: The rate at which the wound area changes, is:

$$\frac{dA}{dt}=-4t^{-3} \qquad (22)$$

The wound area A(t) is obtained by integrating equation (22) with respect to t

$$\int\frac{dA}{dt}=\int -4t^{-3}dt$$

$$A(t)=-4\int t^{-3}dt=-4\frac{t^{-3+1}}{-3+1}+C=\frac{2}{t^2}+C \qquad (23)$$

The fixed wound area C is obtained by inserting A=2 and t=1 in equation (23):

$$2=\frac{2}{1}+C\Rightarrow C=0$$

Put C=0 in equation (23) to obtain the total wound area:

$$A(t)=\frac{2}{t^2} \qquad (24)$$

The specific wound area with in 10 days is obtained by putting t=10 in equation (24):

$$A(10)=\frac{2}{10^2}=\frac{2}{100}=\frac{1}{50}=0.02 \text{ Square centimeters.}$$

Exercise 5.1

1. Evaluate the following indefinite integral and check the result through differentiation:

a. $\int(x^4+3x^3-7)dx$ b. $\int\frac{1}{x^3}dx$ c. $\int(2x^4+x^{\frac{-2}{3}}-x^{\frac{-5}{3}})dx$

d. $\int(1+3t)t^3dt$ e. $\int(t^2-1)^2dt$ f. $\int\frac{x^3+1}{x^3}dx$

g. $\int z^2\sqrt{z}dz$

2. Evaluate the following indefinite integrals by method of substitution:

a. $\int(3x+4)^8dx$ b. $\int 3x^2(x^3-4)dx$ c. $\int(3x^2+7)(x^3+7x)^8dx$

d. $\int\frac{3x^2-7}{(x^3-7x)^4}dx$ e. $\int\frac{x+3x^2}{\sqrt{x}}dx$ f. $\int\frac{x+1}{(x^2+2x+2)}dx$

3. Evaluate the following indefinite integrals by method of substituation:

a. $\int 6e^{6t}dt$ b. $\int xe^{(5x^2+1)}dx$

c. $\int(x^2-2)e^{(x^3-6x+4)}dx$ d. $\int 8^{(7-3x^2)}(-6x)dx$

4. Find the equation of the particular curve that has a slope $4x^3+6x^2$ at a point (1, 0).

5. A certain curve has a slope $x(2x^2-1)^2$ that passes through the point (3, 3). What is the equation of the specific curve?

6. A certain curve has a slope $x\sqrt{2x^2-1}$ that passes through the point (3, 3). What is the equation of the specific curve?

7. For an average person, the rate of change of weight W (in pounds) with respect to height h (in inches) is approximately by

$$\frac{dW}{dh}=0.0015h^2$$

a. Find W(h), if the weight is W=108 pounds in response of height h=60 inches.
b. Find the weight W of a person who is 5 feet 10 inches (=h) tall.

8. The rate of growth of the population N(t) of a newly incorporated city t years after incorporation is estimated to be

$$\frac{dN}{dt}=400+600\sqrt{t},\ 0\leq t\leq 9$$

If the population was 5,000 at the time incorporation, find the population 9 years later.

- ***Method of substitution related to trigonometric functions***

When we want to evaluate the integral, it can be matched with one in the chart exactly. What happens if the matching of the given integral with one of the formulas in the chart is not possible? In this situation. a substitution will be used to reduce the given integral into one that corresponds to a box entry:

1. $\int \sin x dx = -\cos x + C$.	2. $\int \cos x dx = \sin x + C$
3. $\int \tan x dx = -\ln\lvert\cos x\rvert + C$	4. $\int \cot x dx = \ln\lvert\sin x\rvert + C$
5. $\int \sec^2 x dx = \tan x + C$	6. $\int \sec x \tan x \, dx = \sec x + C$
7. $\int \operatorname{cosec}^2 x dx = -\cot x + C$	8. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$
9. $\int \sec x dx = \ln\lvert\sec x + \tan x\rvert + C$	10. $\int \operatorname{cosec} x \, dx = \ln\lvert\operatorname{cosec} x - \cot x\rvert + C$

Example 5.3.4: Evaluate the following integrals;

a. $\int (\cos x)^{\frac{1}{3}} \sin x dx$ b. $\int e^x \sin e^x dx$ c. $\int 2\tan(x+3)dx$ d. $\int 3\operatorname{cosec}^2 3x dx$

Solution:

a. We need to substitute a new variable u(x):

$$\cos x = u, \frac{d}{dx}(\cos x) = \frac{du}{dx} \Rightarrow -\sin x = \frac{du}{dx} \Rightarrow \sin x dx = -du \qquad (25)$$

Substitute (25) in the given integral to obtain

$$\int(\cos x)^{\frac{1}{3}}\sin x dx = -\int u^{\frac{1}{3}}du$$

$$= -\frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1}+C$$

$$= -\frac{3}{4}u^{\frac{4}{3}}+C = -\frac{3}{4}(\cos x)^{\frac{4}{3}}+C, \quad u=\cos x$$

b. We need to substitute a new variable say u(x):

$$e^x = u, \ \frac{d}{dx}(e^x) = \frac{du}{dx} \Rightarrow e^x = \frac{du}{dx} \Rightarrow e^x dx = du \qquad (26)$$

Substitute (26) in the given integral to obtain

$$\int e^x \sin e^x dx = \int \sin u du$$

$$= -\cos u + C = \cos e^x + C, \quad u = e^x$$

c. We need to substitute a new variable say, u(x)

$$(x+3) = u, \frac{d}{dx}(x+3) = \frac{du}{dx} \Rightarrow 1 = \frac{du}{dx} \Rightarrow dx = du \qquad (27)$$

Substitute (27) in the given integral to obtain

$$\int 2\tan(x+3)dx = \int 2\tan u du$$

$$= -2\ln\cos u + C = -2\ln\cos(x+3)+C, \quad u=(x+3)$$

d. We need to substitute a new variable u(x):

$$3x = u, \ \frac{d}{dx}(3x) = \frac{du}{dx} \Rightarrow 3 = \frac{du}{dx} \Rightarrow dx = \frac{du}{3} \qquad (28)$$

Substitute (28) in the given integral to obtain:

$$\int 3\operatorname{cosec}^2 3x dx = \frac{3}{3}\int \operatorname{cosec}^2 u du$$

$$= -\cot u + C = -\cot 3x + C, \quad u = 3x$$

- ***Method of substitution related to inverse trigonometric functions***

General Indefinite Integral Formulae:

11. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}+C$

12. $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$

13. $\int \frac{1}{|x|\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|+C$

Example 5.3.5: Evaluate the following integrals:

a. $\int \frac{dx}{\sqrt{5-4x-x^2}}$ b. $\int \frac{dx}{\sqrt{e^{2x}-4}}$

Solution:

a. The given integral after completing square in denominator, is

$$\int \frac{dx}{\sqrt{5-4x-x^2}}dx = \int \frac{dx}{\sqrt{5+4-4-4x-x^2}}dx = \int \frac{dx}{\sqrt{9-(2+x)^2}}dx \qquad (29)$$

We need to substitute a new variable $u\,(x)$:

$$2+x=u,\ \frac{d}{dx}(2+x)=\frac{du}{dx} \Rightarrow 1=\frac{du}{dx} \Rightarrow dx=du \qquad (30)$$

Substitute (30) in (29) to obtain:

$$\int \frac{dx}{\sqrt{9-(2+x)^2}} = \int \frac{du}{\sqrt{9-u^2}}$$

$$= \sin^{-1}\frac{u}{3}+C = \sin^{-1}\frac{2+x}{3}+C,\ u=2+x$$

b. We need to substitute a new variable $u\,(x)$:

$$e^x=u,\ \frac{d}{dx}(e^x)=\frac{du}{dx} \Rightarrow e^x=\frac{du}{dx} \Rightarrow dx=\frac{du}{e^x}=\frac{du}{u} \qquad (31)$$

Substitute (31) in the given integral to obtain:

$$\int \frac{dx}{\sqrt{e^{2x}-4}} = \int \frac{du}{u\sqrt{u^2-(2)^2}}, \qquad a=2$$

$$= \frac{1}{2}\sec^{-1}\frac{u}{2}+C = \frac{1}{2}\sec^{-1}\frac{e^x}{2}+C, \quad u=e^x$$

ii) Evaulate integrals of the integrands through method of substitution

In this section, we substitute trigonometric functions in place of variables to help integrate algebraic functions. To do this, we use integrals that involve expressions of the form $\sqrt{a^2-x^2}, \sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$, where a is positive constant.

We consider an example of the form:

$$\int \frac{\sqrt{9x^2-1}}{x} dx$$

We substitute $u = 9x^2 - 1$ that on differentiation becomes $du = 18x\,dx$. This integral and many others, cannot be evaluated by substitution. The special trigonometric substitution that are classified as under

1. For $\sqrt{a^2-x^2}$, we need to substitute $x = a\sin\theta,\ a > 0$.
2. For $\sqrt{a^2+x^2}$, we need to substitute $x = a\tan\theta$.
3. For $\sqrt{x^2-a^2}$, we need to substitute $x = a\sec\theta$.

- **Integrals of the form $\int \frac{dx}{a^2-x^2}$, $\int \sqrt{a^2-x^2}\,dx$, $\int \frac{dx}{\sqrt{a^2-x^2}}$**

Example 5.3.6:[Integrand $1/(a^2-x^2)$]: Evaluate the integral $\int \frac{dx}{4-x^2}$.

Solution: Formula-13 for a=2 is used to to obtain the integral of the required integrand:

$$\int \frac{dx}{4-x^2} = \frac{1}{2(2)} \ln\left|\frac{2+x}{2-x}\right| + C = \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + C$$

The substitution of $x = 2\sin\theta$ is not appropriate for this integral. This result can only be obtained by partial fractions.

Example 5.3.7:[Integrand $1/\sqrt{a^2-x^2}$]: Evaluate the integral $\int \frac{dx}{\sqrt{4-x^2}}$.

Solution: We need to use the trigonometric substitution:

$$x = 2\sin\theta,\ dx = 2\cos\theta d\theta \qquad (32)$$

Substitute (32 in the given integral to obtain:

$$\int \frac{dx}{\sqrt{4-x^2}}dx = \int \frac{2\cos\theta d\theta}{\sqrt{4-4\sin^2\theta}}$$

$$= \frac{2}{2}\int \frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{\cos\theta}{\cos\theta}d\theta, \qquad 1-\sin^2\theta = \cos^2\theta$$

$$= \int d\theta = \theta + C = \sin^{-1}\frac{x}{2} + C$$

- $\int \frac{dx}{a^2+x^2}$, $\int \sqrt{a^2+x^2}dx$, $\int \frac{dx}{\sqrt{a^2+x^2}}$

Example 5.3.8: [Integrand $1/(a^2+x^2)$]: Evaluate the integral $\int \frac{dx}{4+x^2}$.

Solution: We need to use the trigonometric substitution:

$$x = 2\tan\theta,\ dx = 2\sec^2\theta d\theta \qquad (33)$$

Substitute (33) in the given integral to obtain:

$$\int \frac{dx}{4+x^2} = \int \frac{2\sec^2\theta d\theta}{4+4\tan^2\theta}$$

$$= \int \frac{2\sec^2\theta d\theta}{4(1+\tan^2\theta)} = \int \frac{2\sec^2\theta d\theta}{4\sec^2\theta} = \frac{1}{2}\int d\theta + C = \frac{\theta}{2} + C = \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

Example 5.3.9: [Integrand $\sqrt{a^2+x^2}$]: Evaluate the integral $\int \sqrt{4+x^2}dx$.

Solution: The substitution of $x = a\tan\theta$ is not appropriate for this integral. This result can only be found by parts integration.

Example 5.3.10:[Integrand $1/\sqrt{x^2+a^2}$]: Evaluate the integral $\int \frac{dx}{x\sqrt{x^2+4}}dx$.

Solution: We need to use the trigonometric substitution:

$$x = 2\tan\theta,\ dx = 2\sec^2\theta d\theta \qquad (34)$$

Substitute (34) in the given integral to obtain:

$$\int \frac{dx}{x\sqrt{x^2+4}}dx = \int \frac{2\sec^2\theta d\theta}{2\tan\theta\sqrt{4\tan^2\theta+4}}$$
$$= 2\ \frac{\sec^2\theta d\theta}{4\tan\theta\sec\theta}$$
$$= \frac{1}{2}\int \frac{\sec\theta}{\tan\theta}d\theta \qquad (35)$$
$$= \frac{1}{2}\int \frac{d\theta}{\sin\theta}$$
$$= \frac{1}{2}\int \operatorname{cosec}\theta d\theta = \frac{1}{2}\ln|\operatorname{cosec}\theta - \cot\theta| + C$$

We know that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \sin\theta(\sec\theta) = \sin\theta(\frac{\sqrt{x^2+4}}{2})$$
$$\frac{x}{2} = \sin\theta\frac{\sqrt{x^2+4}}{2},\quad \tan\theta = \frac{x}{2}$$
$$\sin\theta = \frac{x}{\sqrt{x^2+4}} \Rightarrow \operatorname{cosec}\theta = \frac{\sqrt{x^2+4}}{x}$$
$$x = 2\tan\theta \ \Rightarrow \tan\theta = \frac{x}{2} \Rightarrow \cot\theta = \frac{2}{x}$$

Use these substitutions in integral (35) to obtain:

$$\int \frac{dx}{x\sqrt{x^2+4}}dx = \frac{1}{2}\ln|\operatorname{cosec}\theta - \cot\theta| + C = \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}-2}{x}\right| + C$$

- $\int \frac{dx}{x^2-a^2}$, $\int \sqrt{x^2-a^2}dx$, $\int \frac{dx}{\sqrt{x^2-a^2}}$,

Example 5.3.11: [Integrand $1/(x^2-a^2)$]: Evaluate the integral $\int \frac{dx}{x^2-4}$.

Solution: Formula-14 for a=2 is used to to obtain the integral of the required integrand:

$$\int \frac{dx}{x^2-4} = \frac{1}{2(2)} \ln\left|\frac{x-2}{x+2}\right| + C = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

The substitution of $x = 2\sec\theta$ is not appropriate for this integral. This result can only be obtained by partial fractions.

Example 5.3.12: [Integrand $\sqrt{x^2-a^2}$]: Evaluate the integral $\int\sqrt{x^2-4}\,dx$.

Solution: The substitution of $x = a\sec\theta$ is not appropriate for this integral. The result of this integral can only be obtained by "parts integration". Watch "integration by parts" section for this integral.

Example 5.3.13: [Integrand $1/\sqrt{x^2-a^2}$]: Evaluate the integral $\int \frac{xdx}{\sqrt{x^2-4}}$.

Solution: We need to use the trigonometric substitution:

$$x = 2\sec\theta,\ dx = 2\sec\theta\tan\theta d\theta \qquad (36)$$

Substitute (36) in the given integral to obtain

$$\int \frac{xdx}{\sqrt{x^2-4}} dx = \int \frac{2\sec\theta(2\sec\theta\tan\theta)d\theta}{\sqrt{4\sec^2\theta-4}}$$

$$= 4\int \frac{\sec^2\theta\tan\theta d\theta}{2\sqrt{\sec^2\theta-1}}$$

$$= 2\int \frac{\sec^2\theta\tan\theta}{\tan\theta} d\theta, \qquad \sec^2\theta - 1 = \tan^2\theta$$

$$= 2\int \sec^2\theta d\theta + C$$

$$= 2\tan\theta + C$$

$$= \sqrt{x^2-4} + C, \qquad \tan^2\theta = \sec^2\theta - 1 = \frac{x^2}{4} - 1$$

- $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Example 5.3.14: [Integrand $1/(ax^2+bx+c)$]: Evaluate the integral $\int \frac{dx}{x^2+4x+5}$.

Solution: The given integral after completing square in denominator, is

$$\int\frac{dx}{x^2+4x+5}=\int\frac{dx}{x^2+4x+4+1}=\int\frac{dx}{(x+2)^2+1} \quad (37)$$

We need to substitute a new variable u(x):

$$x+2=u,\ \frac{d}{dx}(x+2)=\frac{du}{dx} \Rightarrow 1=\frac{du}{dx} \Rightarrow dx=du \quad (38)$$

Substitute (38) in (37) to obtain

$$\int\frac{dx}{x^2+4x+5}=\int\frac{dx}{(x+2)^2+1}=\int\frac{du}{u^2+1} \quad (39)$$

We need to use the trigonometric substitutions:

$$u=\tan\theta,\ du=\sec^2\theta d\theta \quad (40)$$

Substitute (40) in (39) to obtain:

$$\int\frac{du}{u^2+1}=\int\frac{\sec^2\theta d\theta}{1+\tan^2\theta}$$

$$=\int\frac{\sec^2\theta d\theta}{1+\tan^2\theta}$$

$$=\int\frac{\sec^2\theta d\theta}{\sec^2\theta}$$

$$=\int d\theta+C=\theta+C=\tan^{-1}u+C=\tan^{-1}(x+2)+C$$

Example 5.3.15: [Integrand $1/\sqrt{ax^2+bx+c}$]: Evaluate the integral $\int\frac{dx}{\sqrt{x^2+4x+5}}$.

Solution: The given integral after completing square in denominator, is

$$\int\frac{dx}{\sqrt{x^2+4x+5}}=\int\frac{dx}{\sqrt{x^2+4x+4+1}}=\int\frac{dx}{\sqrt{(x+2)^2+1}} \quad (41)$$

We need to substitute a new variable say, u(x)

$$x+2=u,\ \frac{d}{dx}(x+2)=\frac{du}{dx} \Rightarrow 1=\frac{du}{dx} \Rightarrow dx=du \quad (42)$$

Substitute (42) in (41) to obtain

$$\int \frac{dx}{\sqrt{x^2+4x+5}} = \int \frac{dx}{\sqrt{(x+2)^2+1}} = \int \frac{du}{\sqrt{u^2+1}} \qquad (43)$$

We need to use the trigonometric substitutions

$$u = \tan\theta,\ du = \sec^2\theta d\theta \qquad (44)$$

Substitute (44) in the integral (43) to obtain:

$$\int \frac{du}{\sqrt{u^2+1}} = \int \frac{\sec^2\theta d\theta}{\sqrt{\tan^2\theta+1}} \qquad (45)$$

$$= \int \frac{\sec^2\theta d\theta}{\sec\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$$

We know that

$\sec^2\theta = \tan^2\theta + 1$

$\sec\theta = \sqrt{\tan^2\theta+1} = \sqrt{u^2+1},\ \tan\theta = u$

Use these substitutions in integral (45) to obtain:

$$\int \frac{du}{\sqrt{u^2+1}} dx = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln|\sqrt{u^2+1}+u| + C = \ln|\sqrt{(x+2)^2+1}+(x+2)| + C, \quad u = x+2$$

- $\int \frac{px+q}{ax^2+bx+c},\ \int \frac{px+q}{\sqrt{ax^2+bx+c}}$

Example 5.3.16: [Integrand $\frac{px+q}{\sqrt{ax^2+bx+c}}$]: Evaluate the integral $\int \frac{x+2}{\sqrt{x^2+4x+5}} dx.$

Solution: The given integral after completing square in denominator, is

$$\int \frac{x+2}{\sqrt{x^2+4x+5}} dx = \int \frac{x+2}{\sqrt{x^2+4x+4+1}} dx = \int \frac{x+2}{\sqrt{(x+2)^2+1}} dx \qquad (46)$$

We need to substitute a new variable say, $u\ (x)$

$$x+2 = u,\ \frac{d}{dx}(x+2) = \frac{du}{dx} \Rightarrow 1 = \frac{du}{dx} \Rightarrow dx = du \qquad (47)$$

Substitute (47) in (46) to obtain:

$$\int \frac{x+2}{\sqrt{x^2+4x+5}}dx = \int \frac{x+2}{\sqrt{(x+2)^2+1}}dx = \int \frac{udu}{\sqrt{u^2+1}} \qquad (48)$$

We need to use the trigonometric substitution

$$u = \tan\theta,\ du = \sec^2\theta d\theta \qquad (49)$$

Substitute (49) in (48) to obtain:

$$\int \frac{udu}{\sqrt{u^2+1}}du = \int \frac{\tan\theta \sec^2\theta d\theta}{\sqrt{\tan^2\theta+1}}$$

$$= \int \frac{\tan\theta \sec^2\theta d\theta}{\sec\theta} = \int \tan\theta \sec\theta d\theta = \sec\theta + C$$

We know that

$$\sec^2\theta = \tan^2\theta + 1$$

$$\sec\theta = \sqrt{\tan^2\theta+1} = \sqrt{u^2+1},\ \tan\theta = u$$

Use these substitutions in the integral (70) to obtain:

$$\int \frac{udu}{\sqrt{u^2+1}} = \sec\theta + C = \sqrt{u^2+1} + C = \sqrt{(x+2)^2+1} + C,\quad u = x+2$$

Definite Integral Formulae involving $\sqrt{x^2 \pm a^2}$ and $\sqrt{a^2 - x^2}$:

14. $\int \sqrt{x^2 \pm a^2}dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln|x + \sqrt{x^2 \pm a^2}| + C$

15. $\int \frac{1}{\sqrt{x^2 \pm a^2}}dx = \ln|x + \sqrt{x^2 \pm a^2}| + C$

16. $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

17. $\int \frac{\sqrt{a^2 - x^2}}{x}dx = \sqrt{a^2 - x^2} - a\ln|\frac{a + \sqrt{a^2 - x^2}}{x}| + C$

Exercise 5.2

1. Evaluate the following indefinite integrals by method of substitution:

a. $\int \sin^4 x \cos x dx$

b. $\int \frac{\cos x \ln(\sin x)}{\sin x}dx$

c. $\int e^x \sin e^x dx$

d. $\int (t+3)\cos(t+3)^2 dt$

2. Evaluate the following indefinite integrals by method of substitution:

a. $\int \tan 2x \sec 2x dx$ b. $\int 4\sec^2 4x dx$ c. $\int \tan x \sec^2 x dx$

d. $\int (\tan 3x + \sec 3x) dx$ e. $\int \frac{\cos^2 x}{\operatorname{cosec} x} dx$ f. $\int \frac{\cot\sqrt{x}}{\sqrt{x}} dx$

g. $\int \frac{\sin x - \cos x}{\sin x + \cos x} dx$ h. $\int \frac{\sin x}{3+2\cos x} dx$

3. Use suitable substitutions and tables to evaluate the following indefinite integrals:

a. $\int \frac{dx}{x^2+16}$ b. $\int \frac{\sin x}{\cos^2 x + 1} dx$ c. $\int \frac{dx}{\sqrt{5-2x^2}}$

d. $\int \frac{dx}{\sqrt{e^{2x}-4}}$ e. $\int \frac{2x+5}{x^2+4x+5} dx$ f. $\int \frac{2+x}{\sqrt{4-2x-x^2}} dx$

g. $\int \frac{dx}{x\sqrt{7x^2-5}}$

5.4 Integration by Parts

In previous sections, we learned some of the basic techniques of integration to solve problems like $\int x^2 dx$ and $\int \sin x dx$. But, how do we evaluate an integral whose integrand is the product of two functions such as

$\int x \sin x dx$, $\int x e^x dx$, $\int x \ln x dx$

To solve integral of the type like that, we have a technique called **integration by parts.**

i) Recognization of integration by parts formula

For this technique, recall the differentiation of the product of two functions $f(x)$ and $g(x)$ w.r.t x:

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}f(x)$$

$$= f(x)g'(x) + g(x)f'(x) \qquad (50)$$

$$f(x)g'(x) = \frac{d}{dx}[f(x)g(x)] - g(x)f'(x)$$

The integral of (50) with respect to x is giving

$$\int f(x)g'(x)dx = \int \frac{d}{dx}[f(x)g(x)]dx - \int g(x)f'(x)dx$$
$$= f(x)g(x) - \int g(x)f'(x)dx$$

the equation that can be transformed into more convenient form by substituting u=f(x) and $v = g(x)$, $du = f'(x)dx$ and $dv = g'(x)dx$:

$$\int udv = uv - \int vdu \qquad (51)$$

This is the standard form of the **integration by parts** formula. **Before to attempt a question, choose u and dv/dx to obtain du and v.**

Example 5.4.1 [Integration by Parts]: Evaluate the integral $\int xe^x dx$.

Solution: The integral rule (51) with $u = x$ and $\frac{dv}{dx} = e^x$ is used to obtain:

$$\int xe^x dx = xe^x - \int e^x (1)dx, \quad \frac{du}{dx} = 1,\ dv = e^x dx,\ v = e^x$$
$$= xe^x - e^x + C$$

ii) *Evaluate the integrals by method of integration by parts*

- $\int \sqrt{a^2 - x^2}dx$, $\int \sqrt{a^2 + x^2}$, $\int \sqrt{x^2 - a^2}$

Example 5.4.2:[Integration by Parts]: Evaluate the following integrals:

a. $\int \sqrt{a^2 - x^2}dx$ b. $\int \sqrt{a^2 + x^2}dx$

Solution:

a. The given integral is

$$I = \int \sqrt{a^2 - x^2}dx \qquad (52)$$

In this problem, we choose $u = \sqrt{a^2 - x^2}$ and $\frac{dv}{dx} = 1$ to integrate the integrand of (52):

$$I=\int\sqrt{a^2-x^2}dx$$

$$=\sqrt{a^2-x^2}(x)-\int\frac{(x)(-x)}{\sqrt{a^2-x^2}}dx,\ \frac{du}{dx}=\frac{1}{2}(a^2-x^2)^{\frac{1}{2}-1}(-2x),\ dv=1dx,\ v=x$$

$$=x\sqrt{a^2-x^2}+\int\frac{x^2}{\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}+\int\frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}},\qquad \textit{add and subtract } a^2$$

$$=x\sqrt{a^2-x^2}+\int\frac{a^2dx}{\sqrt{a^2-x^2}}-\int\sqrt{a^2-x^2}dx$$

$$=x\sqrt{a^2-x^2}+a^2\int\frac{dx}{\sqrt{a^2-x^2}}-I$$

$$2I=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}+C,\qquad \int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$$

$$I=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{C}{2}$$

b. The given integral is

$$I=\int\sqrt{a^2+x^2}dx \tag{53}$$

In this problem, we choose $u=\sqrt{a^2+x^2}$ and $\frac{dv}{dx}=1$ to integrate the integrand of (53):

$$I=\int\sqrt{a^2+x^2}dx$$

$$=\sqrt{a^2+x^2}(x)-\int\frac{(x)(x)}{\sqrt{a^2+x^2}}dx,\ \frac{du}{dx}=\frac{1}{2}(a^2+x^2)^{\frac{1}{2}-1}(2x),\ dv=1dx,\ x=v$$

$$=x\sqrt{a^2+x^2}-\int\frac{x^2}{\sqrt{a^2+x^2}}dx$$

$$=x\sqrt{a^2+x^2}-\int\frac{(a^2+x^2)-a^2}{\sqrt{a^2+x^2}},\qquad \textit{add and subtract } a^2$$

$$= x\sqrt{a^2+x^2} - \int \frac{a^2 dx}{\sqrt{a^2+x^2}} + \int \sqrt{a^2+x^2}dx$$

$$= x\sqrt{a^2+x^2} - a^2\int \frac{dx}{\sqrt{a^2+x^2}} + I$$

$$2I = x\sqrt{a^2+x^2} - a^2 \ln\left|x+\sqrt{a^2+x^2}\right| + C, \quad \int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|x+\sqrt{a^2+x^2}\right|$$

$$I = \frac{x\sqrt{a^2+x^2}}{2} - \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + \frac{C}{2}$$

iii) Evaluate integrals through integration by parts

Example 5.4.3:[Integration by Parts]: Evaluate the integral $\int x \ln x dx$.

Solution: The integral rule (51) with substitution $u = \ln x$ and $dv/dx = x$ is used to obtain:

$$\int x \ln x dx = \ln x\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2}\left(\frac{1}{x}\right)dx, \quad \frac{du}{dx} = \frac{1}{x}, \ \frac{dv}{dx} = x, \ v = \frac{x^2}{2}$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x dx + C$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C \qquad (54)$$

Example 5.4.4:[Repeated use of Integration by Parts]: Evaluate the integral $\int e^x \sin x dx$.

Solution: The integral is

$$I = \int e^x \sin x dx \qquad (55)$$

The integral rule (51) with substitution $u = e^x$ (let u be either e^x or $\sin x$) and $dv/dx = \sin x$ is used to obtain:

$$I = \int e \ \sin x dx = e^x(-\cos x) - \int(-\cos x)e^x dx, \quad \frac{du}{dx} = e^x, \ \frac{dv}{dx} = \sin x \qquad (56)$$
$$= -e^x \cos x + \int e^x \cos x dx$$

It appears that we have not made any progress since we cannot evaluate the new integral. However, the form of the new integral prompts us to apply the technique a second time and see what happens.

Again, the integral of the integral part of equation (56) with substitution $u = e^x$ and $dv/dx = \cos x$ is used in (56) to obtain:

$$I = \int e^x \sin x dx = -e^x \cos x + \left[\int e^x cosxdx\right]$$

$$= -e^x \cos x + \left[e^x \sin x - \int (\sin x)(e^x)dx\right] + C, \frac{du}{dx} = e^x, \frac{dv}{dx} = cosx$$

$$I = -e^x \cos x + e^x \sin x - \int e^x \sin x dx + C$$

$$I = -e^x \cos x + e^x sinx - I + C$$

$$2I = e^x(\sin x - \cos x) + C$$

$$I = \frac{e^x(\sin x - \cos x)}{2} + \frac{C}{2}$$

5.5 Integration by Partial Fractions

Partial fraction decomposition has great value as a tool for integration. This process may be thought of as the "reverse" of adding fractional algebraic expressions, and it allows us to break up rational expressions into simpler terms.

Partial fraction decomposition is an algebraic procedure for expressing a reduced rational function as a sum of fractional parts. For example, the rational expression

$$f(x) = \frac{P(x)}{D(x)} \tag{57}$$

can be decomposed into partial fractions only if P and D have no common factors and if the degree of P is less than the degree of D. If the degree of P is greater than or equal to the degree of D, then use division to obtain a polynomial plus a proper fraction. For example, the rational function after division is:

$$\frac{x^4 + 2x^3 - 4x^2 + x - 3}{x^2 - x - 2} = x^2 + 3x + 1 + \frac{8x - 1}{x^2 - x - 2} \tag{58}$$

$x^2 + 3x + 1$ is our polynomial term

$\frac{8x - 1}{x^2 - x - 2}$ is our proper fraction (this is the part which requires decomposition into partial fractions).

In algebra, the theory of equations tells us that any polynomial P with real coefficients can be expressed as a product of linear and irreducible quadratic powers, some of which may be repeated. This fact can be used to justify the following general procedure for obtaining the partial fraction decomposition of a rational function.

Let $f(x)=\dfrac{P(x)}{D(x)}$, where P(x) and D(x) have no common factors and $D(x)\neq 0$.

The steps involved in decomposing the rational function are the following:

1. If the degree of P is greater than or equal to the degree of D, use long division to express $P(x)/D(x)$ as the sum of a polynomial and a fraction $R(x)/D(x)$ in which the degree of the remainder polynomial R(x) is less than the degree of the denominator polynomial D(x)
2. Factorize the denominator D (x) into the product of linear and irreducible quadratic powers.
3. Express P (x)/D (x) as a cascading sum of partial fractions of the form

$$\frac{A_i}{(x-r)^n} \quad \text{and} \quad \frac{A_j+B_k}{(x^2+sx+t)^m}$$

Verify that the number of constants used is identical to the degree of the denominator.

Example 5.5.1: Evaluate the following integrals:

a. $\displaystyle\int\frac{8x-1}{x^2-x-2}dx$ b. $\displaystyle\int\frac{x^2-6x+3}{(x-2)^3}dx$ c. $\displaystyle\int\frac{2x^3+x^2+2x+4}{(x^2+\text{[illegible]})}dx$

Solution:

a. The integrand is a proper fraction, so we start by factoring the denominator

$x^2-x-2=(x-2)(x+1)$

The denominator factors are the two distinct linear factors, so we can set the rational function equal to the sum of the two partial fractions

$$\frac{8x-1}{x^2-x-2}=\frac{A_1}{x-2}+\frac{A_2}{x+1} \qquad (59)$$

To determine the constants A_1 and A_2, we multiply both sides of the equation (59) by $(x-2)(x+1)$ to obtain:

$$8x-1 = A_1(x+1)+A_2(x-2) \qquad (60)$$

Set $x-2=0 \Rightarrow x=2$ in equation (60) to obtain:

$$8(2)-1 = A_1(2+1)+A_2(2-2) \Rightarrow 15 = 3A_1 \Rightarrow A_1 = 5$$

Set x+1=0 => x = -1 in equation (60) to obtain :

$$8(-1)-1 = A_1(-1+1)+A_2(-1-2) \Rightarrow -9 = -3A_2 \Rightarrow A_2 = 3$$

Use these constants values in equation (59) to obtain:

$$\frac{8x-1}{x^2-x-2} = \frac{A_1}{x-2}+\frac{A_2}{x+1} = \frac{5}{x-2}+\frac{3}{x+1}$$

Use this decomposition instead of rational expression in the given integral to obtain:

$$\int\frac{8x-1}{x^2-x-2}dx = \int\left[\frac{5}{x-2}+\frac{3}{x+1}\right]dx$$
$$= \int\frac{5}{x-2}dx+\int\frac{3}{x+1}dx$$
$$= 5\ln(x-2)+3\ln(x+1)+\ln C$$
$$= \ln(x-2)^5+\ln(x+1)^3+\ln C = \ln C(x-2)^5(x+1)^3$$

b. The integrand is a proper fraction, so we start by factoring the denominator

$$(x-2)^3 = (x-2)(x-2)(x-2)$$

The denominator factors are the three repeated linear factors, so we can set the rational function equal to the sum of the three partial fractions

$$\frac{x^2-6x+3}{(x-2)^3} = \frac{A_1}{x-2}+\frac{A_2}{(x-2)^2}+\frac{A_3}{(x-2)^3} \qquad (61)$$

To determine the constants A_1, A_2 and A_3, we multiply both sides of the equation (61) by $(x-2)^3$ to obtain:

$$x^2-6x+3 = A_1(x-2)^2+A_2(x-2)+A_3$$
$$= A_1(x^2-4x+4)+A_2(x-2)+A_3 \qquad (62)$$

Set $x-2=0 \Rightarrow x=2$ in equation (62) to obtain:

$(2)^2-6(2)+3=A_1(2-2)^2+A_2(2-2)+A_3 \Rightarrow -1=A_3 \Rightarrow A_3=-1$

For constants A_1, A_2, equate the coefficients of x^2 and x on each side of equation (84) to obtain:

$1=A_1$ $\quad x^2$ terms

$-6=-4A_1+A_2$ $\quad x$ terms

Solving this system of equations for the unknowns A_1 and A_2 to obtain

$A_1=1$ and $A_2=-2$

Use these constants values in equation (61) to obtain:

$$\frac{x^2-6x+3}{(x-2)^3}=\frac{A_1}{x-2}+\frac{A_2}{(x-2)^2}+\frac{A_3}{(x-2)^3}=\frac{1}{x-2}-\frac{2}{(x-2)^2}-\frac{1}{(x-2)^3}$$

Use this decomposition instead of rational expression in the given integral. to obtain:

$$\int\frac{x^2-6x+3}{(x-2)^3}dx=\int\left[\frac{1}{x-2}-\frac{2}{(x-2)^2}-\frac{1}{(x-2)^3}\right]dx$$

$$=\ln(x-2)-2\frac{(x-2)^{-2+1}}{-2+1}-\frac{(x-2)^{-3+1}}{-3+1}+C$$

$$=\ln(x-2)+\frac{2}{(x-2)}+\frac{1}{2(x-2)^2}+C$$

c. The intergand is a proper fraction and the denominator factors are the two repeated quadratic factors, so we can set the rational function equal to the sum of the two partial fractions:

$$\frac{2x^3+x^2+2x+4}{(x^2+1)^2}=\frac{A_1x+B_1}{(x^2+1)^2}+\frac{A_2x+B_2}{(x^2+1)} \qquad (63)$$

To determine the constants values, the similar procedure is used to obtain $A_1=0, A_2=2, B_1=3, B_2=1$. With these substitutions, the equation (63) becomes:

$$\frac{2x^3+x^2+2x+4}{(x^2+1)^2}=\frac{3}{(x^2+1)^2}+\frac{2x+1}{(x^2+1)} \qquad (64)$$

Integrate this decomposition to obtain:

$$\int\frac{2x^3+x^2+2x+4}{(x^2+1)^2}dx=\int\frac{3}{(x^2+1)^2}dx+\int\frac{2x+1}{(x^2+1)}dx$$

$$\int\frac{2x^3+x^2+2x+4}{(x^2+1)^2}dx=\int\frac{3}{(x^2+1)^2}dx+\int\frac{2x}{(x^2+1)}dx+\int\frac{1}{(x^2+1)}dx$$

Now the readers are in position, how to find the complete solution of the question.

Hint: $\int\frac{1}{(x^2+1)}dx=\tan^{-1}x$, $\int\frac{2x}{(x^2+1)}dx=\ln u, u=x^2+1$

$\int\frac{3}{(x^2+1)^2}dx=?,\ x=\tan\theta, dx=\sec^2\theta$

Exercise 5.3

1. Evaluate the indefinite integrals after decomposing the following rational functions into partial fractions:

a. $\frac{1}{x(x-3)}$ b $\frac{3x^2+2x-1}{x(x+1)}$ c. $\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}$

d. $\frac{1}{x^3-1}$ e. $\frac{x^4-x^2+2}{x^2(x-1)}$

2. Evaluate the following integrals through partial fractions decomposition:

a. $\int\frac{dx}{x^2-1}$ b. $\int\frac{3x+5}{x^2+2x-3}dx$

c. $\int\frac{-x-3}{2x^2-x-1}dx$ d. $\int\frac{x^2-1}{x^2-2x-15}dx$

e. $\int\frac{x^2}{(x+1)^3}dx$ f. $\int\frac{x}{(x+1)(x^2+1)}dx$

g. $\int\frac{x^2+2}{(x^2+1)^2}dx$ h. $\int\frac{x^4+1}{x^4-1}dx$

3. Use integration by parts to evaluate the following integrals.

a. $\int x e^x dx$ b. $\int x \sin x dx$

c. $\int \tan^{-1} x dx$ d. $\int \sin^{-1} x dx$

e. $\int x^2 (x-3)^{11} dx$ f. $\int e^x \cos x dx$

g. $\int (x+\sin x)^2 dx$ h. $\int e^{2x}\sqrt{1-e^x} dx$

i. $\int x \sin x \cos x \, dx$ j. $\int x^2 \ln x \, dx$

4. Integrate the following integrals by parts integration through appropriate substitution.

a. $\int \frac{\ln x \sin(\ln x)}{x} dx$ b. $\int [\sin 2x \ln(\cos x)] dx$

c. $\int e^{2x} \sin e^x dx$

5. The rate at which the body eliminates a drug (in milliliters per hour) is given by

$$\frac{dR(t)}{dt} = \frac{60t}{(t+1)^2(t+2)}$$

where t is the number of hours since the drug was administered. If $R(0)=0$ is the current drug elimination, how much of the drug is eliminated during the first hour after it was administered? The fourth hour, after it was administered?

6. The rate of change of the voting population of a city with respect to time t (in years) is estimated to be

$$\frac{dN}{dt} = \frac{100t}{(1+t^2)^2}$$

where N(t) is in thousands. If N(0) is the current voting population, then how much will this population increase during the next 3 years?

7. An oil tanker aground on a reef is leaking oil and producing an oil slick that is radiating outward at a rate approximated by

$$\frac{dr}{dt} = \frac{100}{\sqrt{t^2+9}}, \; t \geq 0$$

where t is the radius (in feet) of the circular slick after t minutes. Find the radius of the slick after 4 minutes if the radius is $r = 0$ when $t = 0$.

8. After a person takes a pill, the drug contained in the pill is assimilated into the bloodstream. The rate of assimilation t minutes after taking the pill is

$$\frac{dr}{dt} = te^{-0.2t}, \; r(0) = 0$$

Find the total amount of the drug that is assimilated into the bloodstream during the first 10 minutes after the pill is taken.

5.6 Definite Integrals

Before the definite integral, we need to develop the concept of the area and the area under a curve.

We can determine the area of a figure whose sides are straight lines relatively easily. We can do so equally easily if the figure is a circle, parabola, or ellipse. For any of these figures we use special formulas to determine the area. For example, to determine the surface area of an elliptical swimming pool, we use the formula $A = \pi ab$, where a and b are the lengths of the semiaxes.

At the beginning of this unit we learned that one of the applications of integration is determining the area under a curve-a more difficult area to find. In fact, prior to the development of calculus, the area under a curve only could be approximated. However, the process used to approximate the area led to a technique for determining the exact area.

i) *The area under a Curve*

A technique for approximating the area of a region is to construct rectangles in the region and then take the sum of the areas of the rectangles.

For example, consider the region bounded by the curves $f(x) = x^2$ and the $x=0$ (y-axis), $x = 2$ (y-axis), and $y = 0$ (x-axis) in figure (5.1). To approximate the area of the region, we divide the region in half and construct two rectangles each of whose upper-right corners is on the curve, as shown in figure (5.2). The width of each rectangle is 1, and the heights are $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$. Thus, the sum of the areas of the rectangles is:

$$(1)f(1) + (1)f(2) = (1)(1) + (1)(4) = 5 \qquad (65)$$

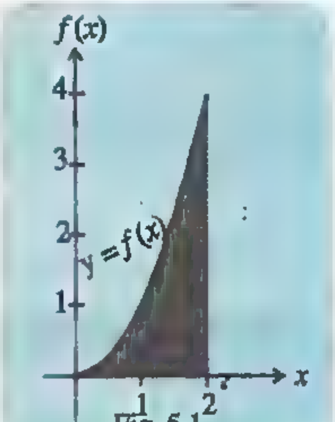


Fig. 5.1

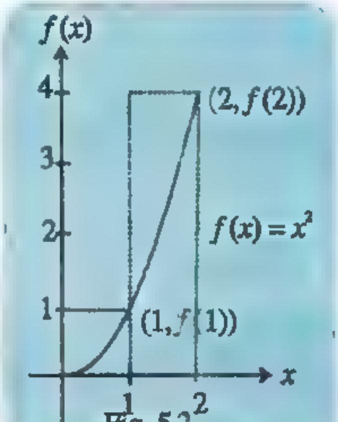


Fig. 5.2

However, from figure (5.2), we can see that the area calculated is larger than the actual area (watch it). To gain a better approximation, we can divide the interval $0 \leq x \leq 2$ into four equal subintervals.

In general, to approximate the area of a region bounded by the curves $y=f(x)$ *(for* $f(x) \geq 0$*)*, and the x-axis; $x = a$ and $x = b$ in the interval $[a, b]$, divide the interval $[a, b]$ into n equal subintervals so that each subinterval has a width of

$$b = a + n\Delta x$$

$$\Delta x = \frac{b-a}{n},$$

and the height of each rectangle is the value of $f(x)$ at the upper-right corner of each rectangle.

The approximate area is the sum of the areas of the rectangles:

$$Area = A \approx f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x. \qquad (66)$$

A summary of the steps is in the box for finding the approximate area of the region:

1. Sketch and label the curve.
2. Determine the width $\Delta x = (b-a)/n$ of each rectangle, where a and b are the endpoints of the interval $[a, b]$ and n is the number of subintervals.
3. $f(x_i)$ is the height of each rectangle (determined by the upper-right corner of each).
4. $Area = A \approx f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x.$

Example 5.6.1: [Approximate Area under a Curve]: Determine the area of the

region bounded by $f(x) = x^2$, x=0, x=2 and the x-axis using n=4 four subintervals.

Solution: The curve is sketched in figure (5.3) with the assumptions:

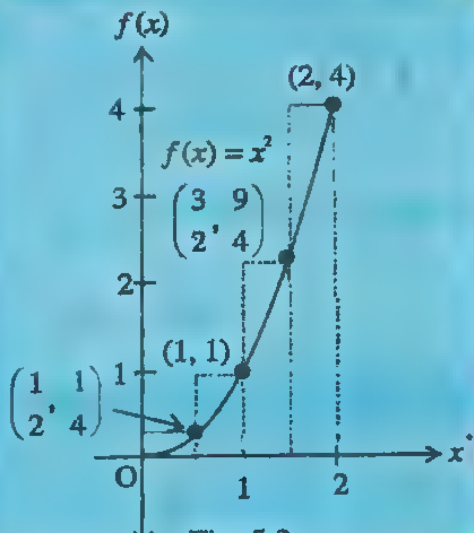


Fig. 5.3

The interval is: $[a, b] = [0, 2]$

The number of subintervals is: $n = 4$

The width of each subinterval is: $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$

The right end points of each subinterval are: $\frac{1}{2}, 1, \frac{3}{2}, 2$

The approximate area is therefore the sum of the areas of the rectangles:

$$A \approx \Delta x f(x_1) + \Delta x f(x_2) + \Delta x f(x_3) + \Delta x f(x_4),\ f(x_i) = x_i^2, i = 0,1,2,3,4$$

$$= \frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f(1) + \frac{1}{2} f\left(\frac{3}{2}\right) + \frac{1}{2} f(2)$$

$$= \frac{1}{2}.\frac{1}{4} + \frac{1}{2}.1 + \frac{1}{2}.\frac{9}{4} + \frac{1}{2}.4$$

$$= \frac{15}{4} = 3\frac{3}{4}$$

The approximate area $3\frac{3}{4}$ square units are larger than the actual area of the region. To get an even more precise approximation of the area, we **increase the number of subintervals** and, thus, **the number of rectangles**.

In general, if we keep increasing the number of rectangles, n becomes larger and the width of each rectangle becomes smaller. Thus, each **result is a** better approximation of the area of the region. If we increase the **number** rectangles to an infinite number, then the result is the actual area of the region, which is $\frac{8}{3}$ square unit.

In light of the above discussion, the approximate area of the region (the sum of the areas of n rectangles) approaches the actual area when $n \to \infty$:

$$A = \lim_{n\to\infty} [\Delta x f(x_1) + \Delta x f(x_2) + \dots + \Delta x f(x_n)],\ \ \Delta x = \frac{b-a}{n} \qquad (67)$$

ii) *Definite integral as the limit of a sum*

This limiting process is what we mean when we say the area is the **definite integral** of $f(x) = x^2$ from x=0 to x=2. It is written symbolically as

$$A = \int_{x=0}^{x=2} x^2 dx = \frac{8}{3} \qquad (68)$$

We read symbol as "the area A equals the integral from $x = 0$ to $x = 2$ of the function $f(x) = x^2$." The number 0 is called the **lower limit of integration**, the number 2 is called the **upper limit of integration**, the function $f(x) = x^2$ is called the **integrand** and the ***dx*** tells us that we are integrating the function $f(x) = x^2$ with respect to the variable x.

Definition 5.6.1 [Definite Integral]: If $f(x)$ is continuous on the interval $[a, b]$ and $[a, b]$ is divided into n equal subintervals whose right-hand points are $x_1, x_2, \dots, x_n$, then the definite integral of $f(x)$ from $x = a$ to $x = b$ is:

$$\int_{x=a}^{x=b} f(x)dx = \lim_{n\to\infty} \frac{(b-a)}{n}[f(x_1) + f(x_2) + \dots + f(x_n)],\ \ \Delta x = \frac{b-a}{n}$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x, \quad i = 1, 2, 3, \dots, n \qquad (69)$$

Example 5.6.2:[Actual area under a curve]: Find the actual area of the region bounded by the curve $f(x) = x^2$ and the x-axis in the interval [0, 2].

Solution: For n subintervals, the width of each rectangle is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

The right end points of the subintervals are $\frac{2}{n}, \frac{4}{n}, \frac{6}{n}, \ldots\ldots, 2.$

Substitute $a = 0, b = 2,\ x_1 = 2/n,\ x_2 = 4/n,\ x_3 = 6/n, \ldots, x_n = 2n/n$ in equation (69) to obtain the actual area:

$$A = \int_0^2 x^2 dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \Delta x\left[f(x_1) + f(x_2) + \ldots\ldots + f(x_n)\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[f\left(\frac{2}{n}\right) + f\left(\frac{4}{n}\right) + f\left(\frac{6}{n}\right) + \ldots\ldots + f(2)\right],\ f(x) = x^2$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2} + \frac{16}{n^2} + \frac{36}{n^2} + \ldots\ldots + \frac{4n^2}{n^2}\right],\ 4 = 4n^2/n^2$$

$$= \lim_{n\to\infty} \frac{8}{n^3}\left[1 + 2^2 + 3^2 + \ldots\ldots + n^2\right]$$

$$= \lim_{n\to\infty} \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \frac{8}{n^3}\left[\frac{2n^3 + 3n^2 + n}{6}\right]$$

$$= \lim_{n\to\infty}\left[\frac{16}{6} + \frac{24}{6n} + \frac{8}{6n^2}\right]$$

$$= \frac{8}{3} + 0 + 0 = \frac{8}{3}$$

Example 5.6.3:[Approximate Area under a Curve]: The rate of change of the annual maintenance charges for a certain machine is shown in the figure (5.4).

Approximate the total maintenance charges over the 10-year life of the machine by using rectangles. In this situation, is it possible to show the actual maintenance charges for a certain machine?

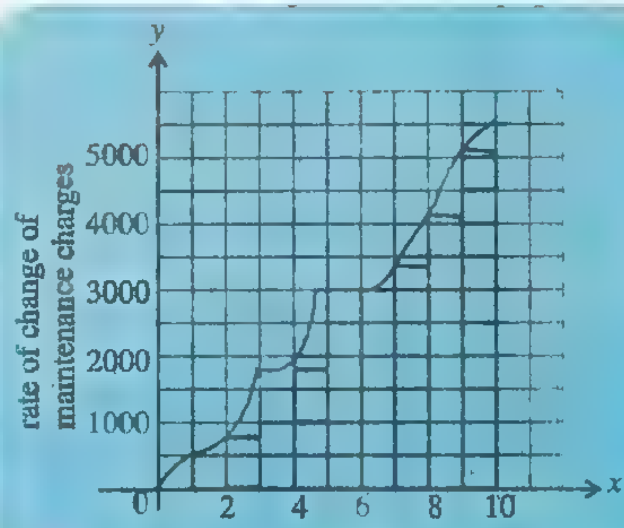


Fig. 5.4.

Solution: From the given figure, we collect the following information

The interval is: $[a, b] = [0, 10]$

The number of subintervals is: $n = 10$

The width of each subinterval is: $\Delta x = \dfrac{b-a}{n} = \dfrac{10-0}{10} = 1$

We choose the left end points of each subinterval that are 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

The total maintenance charges over the 10-year life of the machine are approximated by:

$$M : Chages \approx \Delta x f(x_0) + \Delta x f(x_1) + \Delta x f(x_2) + \Delta x f(x_3) + \ldots\ldots + \Delta x f(x_9)$$
$$= (1)(0) + (1)(500) + (1)(750) + (1)(1800) + (1)(1800) + (1)(3000)$$
$$+ (1)(3000) + (1)(3400) + (1)(4200) + (1)(5200)$$
$$= 23,650$$

Approximately, \$23,650 will be spent on maintenance over the 10-year life of the maintenance.

No, it is not possible to calculate the actual maintenance charges; because of the nonavailability of the rule of charges, that is a function $f(x)$.

Exercise 5.4

1. In each case, determine the approximate area of the region bounded by $f(x) = 2x+1$, $x = a$ and $x = b$ for n subintervals:

a. $n = 2, a = 0, b = 2$ b. $n = 4, a = 0, b = 2$
c. $n = 8, a = 0, b = 2$ d. $n = 2, a = 1, b = 5$

2. In each case, determine the approximate area of the region bounded by $f(x) = x^2 + 1$, $x = a$ and $x = b$ for n subintervals:

a. $n = 2, a = 0, b = 2$ b. $n = 4, a = 0, b = 2$
c. $n = 8, a = 0, b = 2$ d. $n = 2, a = 1, b = 5$

3. In each case, determine the actual value of the integral using definition 5.6.1:

a. $\int_{x=0}^{x=3} 3x\,dx$ b. $\int_{x=0}^{x=3} (2x-4)\,dx$

c. $\int_{x=0}^{x=2} x^2\,dx$ d. $\int_{x=2}^{x=3} (x^2-4)\,dx$

iii) *The Fundamental theorem of integral calculus*

In previous section, we learned that we can determine the area of a region with a definite integral. However, with the tools available to us at this time, evaluating a definite integral using the summation process is rather tedious and time consuming. To provide us with a more efficient method of evaluating the definite integral, we now consider a very important theorem in calculus, the **"fundamental theorem of integral calculus"**. This explanation will show that the definite integral can be applied in a general manner and not only to the concept of area (just as we found that the derivative has wider applications than just finding the slope of a line).

To help provide a better understanding of the meaning of the fundamental theorem of integral calculus, let us begin with area of a region using definite integral

$$Area = \int_{x=a}^{x=b} f(x)dx \qquad (70)$$

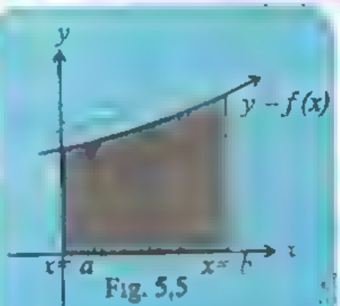


Fig. 5.5

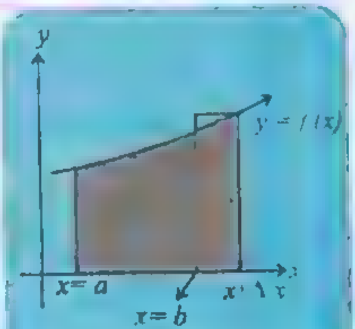


Fig. 5.6

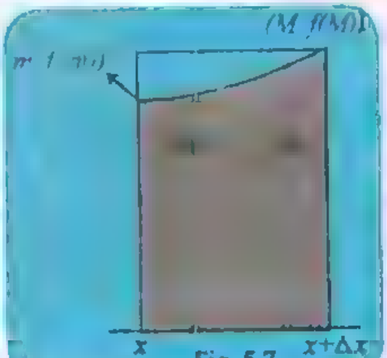


Fig. 5.7

To develop the theorem, we need to introduce a new function called the area function $A(x)$. The function indicates the area of the region under the graph of the function from x=a to x=b in the figure 5.5. The area function $A(x)$ is the area from a to x that must be continuous and non-negative on the interval $[a, b]$.

If we increase x by Δx, then the area $A(x)$ under the curve will increase by an amount that we call ΔA (figure 5.6). We can see that ΔA is slightly bigger than the area of the inscribed rectangle and slightly smaller than the area of the circumscribed rectangle. In figure 5.7, the smaller rectangle is inscribed (within the curve) and the large rectangle is circumscribed

For the area of the inscribed rectangle, we take the minimum value of $f(x)$ within the closed interval $[x, x+\Delta x]$. We call this minimum value f(m). For the area of the circumscribed rectangle, we take the maximum value within the closed interval $[x, x+\Delta x]$. We refer to this value as $f(M)$. Hence the minimum area is $f(m)\Delta x$ and the maximum area is $f(M)\Delta x$.

Algebraically, we can write

$$f(m)\Delta x \le \Delta A \le f(M)\Delta x$$

$$f(m) \le \frac{\Delta A}{\Delta x} \le f(M), \quad \Delta x \ne 0 \qquad (71)$$

If we take the limit as $\Delta x \to 0$, then $f(m)$ and $f(M)$ approach the same point on the curve and both approach $f(x)$

$$f(x) \le \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} \le f(x)$$

which states that

$$\frac{dA}{dx} = f(x), \qquad \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx} \qquad (72)$$

Integrating (72) to obtain

$$A(x) = F(x) + C \qquad (73)$$

Here $F(x)$ is the antiderivative of $f(x)$. To determine a real value of A(x), we must solve equation (73) for C.

Put $x = a$ in (73) to obtain:

$$A(a) = F(a) + C$$
$$0 = F(a) + C, \; A(a) = 0$$
$$C = -F(a)$$

Put $x = b$ in (73) to obtain:

$$A(b) = F(b) + C$$
$$A(b) = F(b) - F(a), \; C = -F(a) \qquad (74)$$

The last equation (74) tells us that if it is possible to find an antiderivative of $f(x)$, then we can evaluate the definite integral $\int_{x=a}^{x=b} f(x)dx$. This is nicely condensed in the fundamental theorem.

Theorem 5.2:[Fundamental Theorem of Integral Calculus]: If a function $f(x)$ is continuous on the closed interval [a,b], then the definite integral of a function f(x) in the interval [a,b] is :

$$\int_{x=a}^{x=b} f(x)dx = \left| F(x) \right|_{x=a}^{x=b} = F(b) - F(a) \qquad (75)$$

Here $F(x)$ is any function such that $F'(x) = f(x)$ for all x in $[a, b]$.

It is important to recognize that the fundamental theorem of integral calculus describes a means for evaluating a definite integral. It does not provide us with a technique for finding the antiderivative. To find the antiderivative of a definite integral, we use the same techniques we used to find the antiderivative of the indefinite integral. But what happens to the constant C? This constant C drops out as illustrated below:

$$\int_{x=a}^{x=b} f(x)dx = \left|F(x)+C\right|_{x=a}^{x=b}$$

$$= \left[(F(b)+C)-(F(a)+C)\right] \quad (76)$$

$$= F(b)-F(a)+C-C = F(b)-F(a)$$

- ***Recognization of basic properties of the definite integrals***

In computations involving integrals, it is often helpful to use the seven basic properties related to fundamental theorem of calculus that are listed below:

Linearity Rule

If f(x) and g(x) are integrable on interval [a, b], then for any constant multiples *r, and s,* the definite integral is defined to be:

$$\int_a^b [rf(x)+sg(x)]dx = r\int_a^b f(x)dx + s\int_a^b g(x)dx$$

Equality Rule

If f(x) and g(y) are integrable on interval [a, b] w.r.t x, and y and f(x) = g(y), then the definite integral of f(x) w.r.t x on the left equals the definite integral of g(y) w.r.t y on the right:

$$\int_a^b f(x)dx = \int_a^b g(y)dy$$

Subdivision Rule

For any number c such that *a* < c < b, the definite integral of f(x) w.r.t x in the interval [a, b] is:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Assume that all three integrals exist.

Dominance Rule

If *f*(*x*) and g(*x*) are integrable on interval [*a*,b] w.r.t x and *f(x)* ≤ *g(x)* throughout this interval, then the definite integral of f(x) and g(x) in the interval [a, b] are:

$$\int_a^b f(x)dx \le \int_a^b g(x)dx$$

Point Rule

If (x) is integrable on interval $[a, b]$ w.r.t x and $a = b$ or $b = a$, then the interval is really just a point and the integral of f(x) on this interval $[a, a]$ is defined to be 0:

$$\int_a^a f(x)dx = 0$$

Opposite Rule

If $f(x)$ is integrable on interval $[a, b]$ w.r.t x and the lower limit b of integration is a larger number than the upper limit a, then the definite integral of f(x) from b to a is the opposite of the definite integral of f(x) from a to b.

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Even and Odd Function Rule

If $f(x)$ is integrable on interval [-a, a] w.r.t x, then for a number 0 in the interval [-a, a], the definite integral of f(x) from –a to a is 2 times the definite integral of f(x) from 0 to a:

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = \int_a^0 f(-x)d(-x) + \int_0^a f(x)dx$$

$$= -\int_a^0 f(-x)dx + \int_0^a f(x)dx$$

$$= \int_0^a f(-x)dx + \int_0^a f(x)dx$$

$$= \int_0^a [f(-x) + f(x)]dx = \begin{cases} 2\int_0^a f(x)dx, & when\ f(-x) = f(x) : Even \\ 0, & when\ f(-x) = -f(x) : Odd \end{cases}$$

iv) *Evaluate definite integrals through basic properties*

Example 5.6.4:[Basic Properties]: Evaluate the following definite integrals:

a. $\int_1^2 (2x^2 + 4x + 1)dx$ b. $\int_1^2 (2x^2 + 4x + 1)dx = \int_1^2 (2y^2 + 4y + 1)dy$

c. $\int_0^2 (x^2+1)dx = -\int_2^0 (x^2+1)dx$

d. $\int_0^2 (x^2+1)dx = \int_0^1 (x^2+1)dx + \int_1^2 (x^2+1)dx$

e. $\int_1^{+1} x^2 dx = \int_{-1}^0 x^2 dx + \int_0^1 x^2 dx = 2\int_0^1 x^2 dx$

Solution:

a. $\int_1^2 \left(2x^2+4x+1\right)dx = \left|\frac{2x^3}{3}+\frac{4x^2}{2}+x\right|_1^2$ *Linearity property*

$$= \left(\frac{16}{3}+\frac{16}{2}+2\right)-\left(\frac{2}{3}+\frac{4}{2}+1\right)$$

$$= \frac{92}{6}-\frac{11}{3}$$

$$= \frac{70}{6} = \frac{35}{3}$$

b. $\int_1^2 (2x^2+4x+1)\,dx = \int_1^2 (2y^2+4y+1)\,dy$, *Equality Rule*

$$\left|\frac{2x^3}{3}+\frac{4x^2}{2}+x\right|_1^2 = \left|\frac{2y^3}{3}+\frac{4y^2}{2}+y\right|_1^2$$

$$\frac{35}{3} = \frac{35}{3}, \text{ since } f(x) = g(y)$$

c. $\int_0^2 (x^2+1)dx = -\int_2^0 (x^2+1)dx$, *Opposite Rule*

$$\left|\frac{x^3}{3}+x\right|_0^2 = -\left|\frac{x^3}{3}+x\right|_2^0$$

$$\frac{8}{3}+2 = -\left(0-(\frac{8}{3}+2)\right)$$

$$\frac{14}{3} = \frac{14}{3}$$

d. $\int_0^2 (x^2+1)dx = \int_0^1 (x^2+1)dx + \int_1^2 (x^2+1)dx$, *Subdivision Rule*

$$\left|\frac{x^3}{3}+x\right|_0^2 = \left|\frac{x^3}{3}+x\right|_0^1 + \left|\frac{x^3}{3}+x\right|_1^2$$

$$\frac{14}{3} = \frac{4}{3}+\frac{8}{3}+2-\left(\frac{1}{3}+1\right)\frac{14}{3} = \frac{14}{3}$$

e. $\int_{-1}^{+1} x^2 dx = \int_{-1}^{0} x^2 dx + \int_0^1 x^2 dx = 2\int_0^1 x^2 dx$, *Odd & Even Functions*

$$\left|\frac{x^3}{3}\right|_{-1}^{1} = \left|\frac{x^3}{3}\right|_{-1}^{0} + \left|\frac{x^3}{3}\right|_0^1 = 2\left|\frac{x^3}{3}\right|_0^1$$

$$\frac{1}{3}+\frac{1}{3} = 0+\frac{1}{3}+\frac{1}{3} = 2\left(\frac{1}{3}\right)$$

$$\frac{2}{3} = \frac{2}{3} = \frac{2}{3}$$

Example 5.6.5 : Evaluate the following definite integrals:

a. $\int_0^1 \frac{2e^{4x}-3}{e^{2x}}dx$ b. $\int_{\pi/3}^{\pi/2} x\sin x^2 dx$

Solution:

a. $\int_0^1 \frac{2e^{4x}-3}{e^{2x}}dx = \int_0^1 \left(2e^{2x}-3e^{-2x}\right)dx$

$$= \left|2\frac{e^{2x}}{2} - 3\frac{e^{-2x}}{-2}\right|_0^1$$

$$= \left|e^{2x}+\frac{3}{2}e^{-2x}\right|_0^1$$

$$= \left(e^2+\frac{3}{2}e^{-2}\right)-\left(e^0+\frac{3}{2}e^0\right)$$

$$= e^2+\frac{3}{2}e^{-2}-1 = 5.092$$

b. We need to substitute a new variable $u(x)$:

$$x^2 = u,\ \frac{d}{dx}(x^2) = \frac{du}{dx} \Rightarrow 2x = \frac{du}{dx} \Rightarrow xdx = \frac{du}{2}$$

The lower and upper limit of $x = \pi/3$ and $x = \pi/2$ are used in $x^2 = u$ to btain the lower and upper limit of u:-

$$x = \pi/3:\ x^2 = u \Rightarrow (\pi/3)^2 = u \Rightarrow \frac{\pi^2}{9} = u$$

$$x = \pi/2\ x^2 = u \Rightarrow (\pi/2)^2 = u \Rightarrow \frac{\pi^2}{4} = u$$

Substitute all these in the given integral to obtain:

$$\int_{\pi/3}^{\pi/2} x\sin x^2 dx = \frac{1}{2}\int_{\pi^2/9}^{\pi^2/4} \sin u\, du$$

$$= \left|\frac{-\cos u}{2}\right|_{\pi^2/9}^{\pi^2/4}$$

$$= -\frac{1}{2}\left(\cos\frac{\pi^2}{4} - \cos\frac{\pi^2}{9}\right)$$

$$= \frac{-1}{2}[\cos(2.4647) - \cos(1.0966)],\ use\ radians$$

$$= \frac{-1}{2}(-0.7812 - 0.4566)$$

$$= \frac{-1}{2}(-1.2378) = 0.6189$$

Definite Integral through Integration-by-Parts

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

Example 5.6.6: [Integration by Parts]: Evaluate the following definite integrals:

a. $\int_0^1 xe^x dx$ b. $\int_0^1 e^x \sin x dx$

Solution:

a. The technique of integration by parts with $u = x$ and $\frac{dv}{dx} = e^x$ is used to obtain:

$$\int_0^1 xe^x dx = \left|xe^x\right|_0^1 - \int_0^1 e^x 1dx, \quad u = x, du = dx, \frac{dv}{dx} = e^x, v = e^x$$

$$= (e^1 - 0) - \left|e^x\right|_0^1 = (e^1 - 0) - (e^1 - e^0) = e^1 - e^1 + e^0 = 1$$

b. The integral is

$$I = \int_0^1 e^x \sin x dx$$

The integration by parts rule with substitution $u = e^x$ and $dv/dx = \sin x$ is used to obtain:

$$I = \int_0^1 e^x \sin x dx$$

$$= \left|e^x(-\cos x)\right|_0^1 - \int_0^1 (-\cos x)e^x dx, \; u = e^x, du = e^x dx, \frac{dv}{dx} = \sin x, \; v = -\cos x$$

$$= -(e^1 \cos 1 - e^0 \cos 0) + \int_0^1 e^x \cos x dx$$

$$= -2.718(0.540) + 1 + \int_0^1 e^x \cos x dx = -0.468 + \int_0^1 e^x \cos x dx, \; use \; radians$$

$$= -0.468 + \int_0^1 e^x \cos x dx, \; Again \; integration \; by \; parts$$

$$= -0.468 + \left|e^x \sin x\right|_0^1 - \int_0^1 (\sin x)(e^x) dx, \; u = e^x, \frac{dv}{dx} = \cos x, \; v = \sin x$$

$$= -0.468 + (e^1 \sin 1 - e^0 \sin 0) - \int_0^1 e^x \sin x dx$$

$$I = -0.468 + (2.718(0.841) - (1)(0)) - I$$

$$2I = -0.468 + 2.287 = 1.819$$

$$I = \frac{1.819}{2} = 0.91$$

v) *Definite integral as the area under a curve*

Definition 5.6.1:[Definite Integral as the Area undera Curve]: If $f(x)$ is continuous and $f(x) \geq 0$ on the closed interval $[a, b]$, then the area under a curve $y = f(x)$ on the interval $[a, b]$ is given by the definite integral of $f(x)$ on $[a, b]$:

$$Area = \int_a^b f(x)dx = F(b) - F(a) \qquad (78)$$

Area between a curve and the x-axis

The steps involved in finding the area between a curve and the x-axis are the following:

1. The definite integralre $\int_a^b f(x)dx$ presents the sum of the signed areas between the graph of y=f(x) and the x-axis from x=a to x=b, where the area above the x-axis (peak) are counted positively and the areas below the x-axis (valley) are counted negatively. This is shown in the figure (5.8):

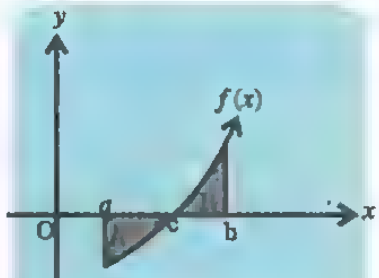


Fig. 5.8

2. If f(x) is a continuous function over the interval [a,b], then the area between y=f(x) and the x-axis from x=a to x=b can be found using definite integrals as follows:

   - For $f(x) \geq 0$ over [a,b], the area is: $Area = \int_a^b [+f(x)]dx$

- For $f(x) \leq 0$ over [a,b], the area is: $Area = \int_a^b [-f(x)]dx$

If f(x) is positive for some values of x and negative for others on an interval (as in figure (5.8)), then, the area between the graph of f(x) and the x-axis can be found by **(dividing the interval into subintervals over which f(x) is always positive or always negative)** taking the sum of the areas of subregions over each subinterval:

$$Area = \int_a^b f(x)dx = \int_a^c [-f(x)dx] + \int_c^b [+f(x)dx] = -A + B \qquad (79)$$

In Fig. 5.8, A represents the area between y=f(x) and the x-axis from x=a to x=c, and B represents the area between y=f(x) and the x-axis from x=c to x=b. Both A and B are positive quantities. Since $f(x) \geq 0$ on the interval [c,b], the area is $\int_c^b [+f(x)]dx = B$ and $f(x) \leq 0$ on the interval [a,c], the area is $\int_a^c [-f(x)dx] = -A$.

vi) Application of definite integral as the area under a curve

Example 5.6.7: [Definite Integral as an Area]: Find the area between the x-axis and the curve $f(x) = x^2 - 4$ from $x = 0$ to $x = 4$.

Solution: First find out the x-intercepts of a curve $f(x) = x^2 - 4$ that can be found by solving the equation of a curve:

$x^2 - 4 = 0 \Rightarrow x = 2, -2$

The subintervals of the interval [0,4] are therefore [-2,0], [0,2] and [2,4]. The total area of the region in the required interval [0,4] is the sum of the areas of the subregions in the subintervals [0,2] and [2,4]:

$$Area = \int_0^2 [-f(x)]dx + \int_2^4 [+f(x)]dx, \; f(x) \leq 0 \, in [0,2] \, and \; f(x) \geq 0 \, in [2,4]$$

$$= -\int_0^2 (x^2 - 4)dx + \int_2^4 (x^2 - 4)dx$$

$$= -\left|\frac{x^3}{3} - 4x\right|_0^2 + \left|\frac{x^3}{3} - 4x\right|_2^4$$

$$= -\left|\frac{8}{3} - 8 - (0 - 0)\right| + \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right) = \frac{16}{3} + \frac{16}{3} - \frac{16}{3} = 16 \textit{ square units}$$

The sketch of the region is shown in the Fig. 5.9. The area over the entire interval [0, 4]

$$A = \int_0^4 (x^2 - 4)dx = x^3 - 4x\Big|_0^4 = \frac{16}{3}$$

is not the correct area. This definite integral does not represent the area over the entire interval [0, 4], but is just a real number.

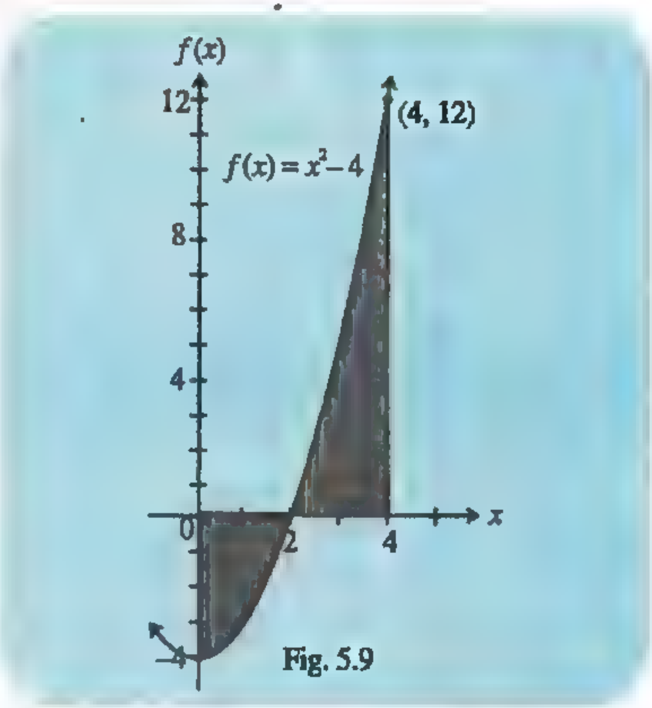


Fig. 5.9

Exlampe 5.6.8:[Definite Integral as an Area]: Find the area between the x-axis and the curve $f(x) = x^2 - 2x$ from x = −1 to x = 3.

Solution: First find out the x-intercepts of a curve $f(x) = x^2 - 2x$ that can be found by solving the equation of a curve:

$x^2-2x=0 \Rightarrow x=0,2$

The subintervals of the interval [-1,3] are therefore [-1,0], [0,2]and [2,3]. The total area of the region in the required interval [-1,3] is the sum of the areas of the subregions in the subintervals [-1,0], [0,2]and [2,3]:

$$A=\int_{-1}^{0}[+f(x)]dx+\int_{0}^{2}[-f(x)]dx+\int_{2}^{3}[+f(x)]dx,\ f(x)\geq 0\ in[-1,0],[2,3]$$

$$=\int_{-1}^{0}(x^2-2x)dx-\int_{0}^{2}(x^2-2x)dx+\int_{2}^{3}(x^2-2x)dx$$

$$=\left|\frac{x^3}{3}-\frac{2x^2}{2}\right|_{-1}^{0}-\left|\frac{x^3}{3}-\frac{2x^2}{2}\right|_{0}^{2}+\left|\frac{x^3}{3}-\frac{2x^2}{2}\right|_{2}^{3}$$

$$=(0-0)-\left(\frac{-1}{3}-1\right)-\left|\left(\frac{8}{3}-4\right)-(0-0)\right|+\left(\frac{27}{3}-9\right)-\left(\frac{8}{3}-4\right)$$

$$=\frac{4}{3}+\frac{4}{3}+\frac{4}{3}=3\left(\frac{4}{3}\right)=4$$

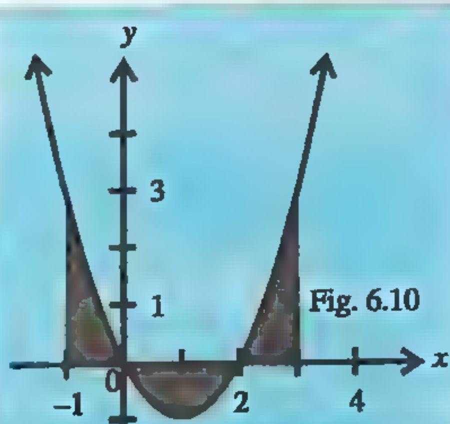


Fig. 6.10

The sketch of the region is shown in the figure (5.10).

Definition 5.6.9:[Definite Integral as a Distance]: The distance S traveled by an object moving with continuous positive rate function r(t) along a straight line from time t = a to t=b is:

$$S=\int_{a}^{b}r(t)dt=F(b)-F(a) \qquad (80)$$

Example:5.6.10:[Definite Integral as a Distance]: An object moves along a straight line with curve $r(t) = t^2$ for $t > 0$. How for does the object travel between times $t = 1$ and $t = 2$?

Solution: The distance S traveled by an object with $r(t) = t^2$ from $a = 1$ to $b = 2$ is:

$$S = \int_a^b r(t)dt = \int_1^2 t^2 dt = \left|\frac{t^3}{3}\right|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Thus, we expect the object to travel 7/3 units during the time interval [1, 2].

Exercise 5.5

1. Evaluate the following definite integrals:

a. $\int_3^4 5dx$ b. $\int_{12}^{20} dx$ c. $\int_1^2 (2x^{-2} - 3)dx$

d. $\int_1^4 3\sqrt{x}dx$ e. $\int_2^3 12(x^2-4)^5 xdx$ f. $\int_{-6}^0 \sqrt{4-2x}dx$

g. $\int_{-1}^7 \frac{x}{\sqrt{x+2}}dx$ h. $\int_0^1 (e^{2x}-2x)^2(e^{2x}-1)dx$

2. Evaluate the following definite integrals:

a. $\int_2^3 x\sqrt{2x^2-3}dx$ b. $\int_0^1 x\sqrt{3x^2+2}dx$

c. $\int_0^1 \frac{x-1}{x^2-2x+3}dx$ d. $\int_{-1}^1 \frac{e^{-x}-e^x}{(e^{-x}+e^x)^2}dx$

3. Evaluate the following definite integrals:

a. $\int_{\pi/2}^{\pi} \cos\left(\frac{x}{2}+\pi\right)dx$ b. $\int_{0.75}^{2.5} x\cos x^2 dx$ c. $\int_0^{\pi/4} \sec^2\theta d\theta$

d. $\int_0^{\pi/4} \tan 2\pi x dx$ e. $\int_0^1 \frac{1}{x^2+1}dx$ f. $\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1}}dx$

4. Evaluate the following definite integrals:

a. $\displaystyle\int_{1}^{2}\frac{5t^2-3t+18}{t(9-t^2)}\,dt$ b. $\displaystyle\int_{2}^{3}\frac{4x^5-3x^4-6x^3+4x^2+6x-1}{(x-1)(x^2-1)}\,dx$

c. $\displaystyle\int_{3}^{5}\frac{x^2-2}{(x-2)^2}\,dx$ d. $\displaystyle\int_{1}^{2}\frac{4}{t^3+4t}\,dt$ e. $\displaystyle\int_{1}^{3}\ln(2x+1)\,dx$

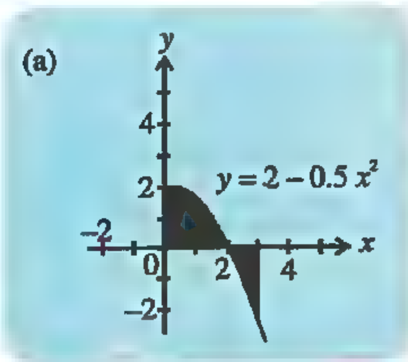
f. $\displaystyle\int_{1}^{4}\frac{\ln x}{x^3}\,dx$ g. $\displaystyle\int_{1}^{2}x(x-1)^6\,dx$ h. $\displaystyle\int_{0}^{1}(x-3)e^x\,dx$

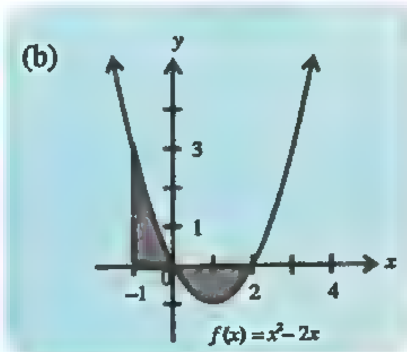
5. Use the definite integral to find out the area between the curve f(x) and the x-axis over the indicated interval [a, b]:

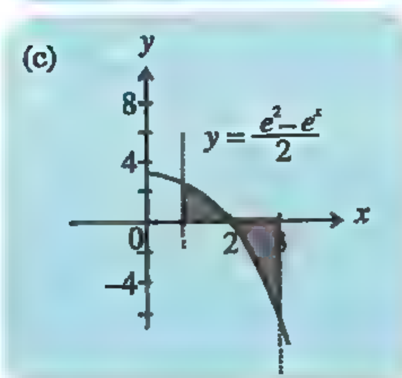
a. $f(x)=4-x^2$, $[0,3]$ b. $f(x)=x^2-5x+6$, $[0,3]$

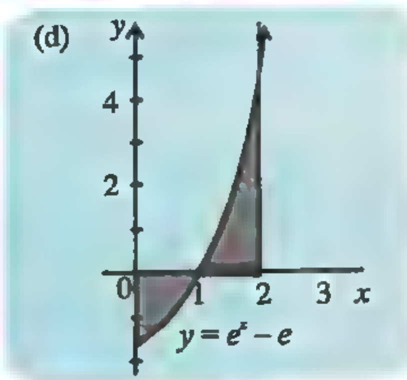
c. $f(x)=x^2-6x+8$, $[0,4]$ d. $f(x)=5x-x^2$, $[1,3]$

6. Set up definite integrals in problems a to d that represent the indicated shaded areas:









7. An oil tanker is leaking oil at a rate given in barrels per hour by

$$\frac{dL}{dt}=\frac{80\ln(t+1)}{(t+1)}$$

Where t is the time in hours after the tanker hits a hidden rock (when t=0).

a. Find the total number of barrels that the ship will leak on the first day.

b. Find the total number of barrels that the ship will leak on the second day.

c. What is happening over the long run to the amount of oil leaked per day?

Glossary

- **Antidertivative:** $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.
- **Indefinite Integral:** If $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, for any real number C.
- **Integration by Parts:** If f(x) and g(x) are integrable functions w.r.t x, then the integral of the product of f(x) and g(x) w.r.t x is:

$$\int u dv = uv - \int v du, \qquad v = g(x),\ du = f'(x)dx \text{ and } dv = g'(x)dx:$$

- **Definite Integral (Actual Area under a Curve):** If $f(x)$ is continuous on the interval $[a, b]$ and $[a, b]$ is divided into n equal subintervals whose right-hand points are $x_1, x_2, \ldots, x_n$, then the definite integral of $f(x)$ from $x = a$ to $x = b$ is:

$$\int_{x=a}^{x=b} f(x)dx = \lim_{n\to\infty} \frac{(b-a)}{n}\left[f(x_1) + f(x_2) + \ldots + f(x_n)\right], \quad \Delta x = \frac{b-a}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x, \quad i = 1, 2, 3, \ldots, n$$

- **Definite Integral through Integration by Parts:** The definite integral of the product of two functions f(x) and g(x) w.r.t x is:

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$$

- **Definite Integral as the Area undera Curve:** If $f(x)$ is continuous and $f(x) \geq 0$ on the closed interval $[a, b]$, then the area under a curve $y = f(x)$ on $[a, b]$ is given by the definite integral of $f(x)$ on $[a, b]$:

$$Area = \int_a^b f(x)dx = F(b) - F(a)$$

- **Fundamental Theorem of Calculus:** If a function $f(x)$ is continuous on the closed interval $[a, b]$, then

$$\int_{x=a}^{x=b} f(x)dx = \left[F(x)\right]_{x=a}^{x=b} = F(b) - F(a)$$

Where $F(x)$ is any function such that $F'(x) = f(x)$ for all x in $[a, b]$.

UNIT 6

PLANE ANALYTIC GEOMETRY STRAIGHT LINE

This unit tells us, how to :

- develop the distance formula between the two points.
- obtain the coordinates of a point that divides the line segment in the ratio m:n.
- show that the medians and angle bisectors of a triangle are concurrent.
- define the slope of a line and the condition at which the two lines are parallel or perpendicular.
- find the equations of lines parallel to x– and y–axis.
- define the intercepts of a line and the different standard forms of a line through some conditions.
- reduce the general form of a line into different standard forms of a line.
- recognize a point that lies above or below the given line.
- find the perpendicular distance from a line to a point.
- find the angle between the two straight lines.
- find the equation of a family of lines passing through the point of intersection of the given two straight lines.
- calculate the angles of the triangle when the slopes of the sides are available.
- find the condition of concurrency of three straight lines.
- find the equations of the medians, altitudes and right bisectors of a triangle.
- show that the three right bisectors, three altitudes and three medians are concurrent.
- find the area of a rectangular region when the vertices are available.
- recognize the homogeneous linear and quadratic equations in two variables.
- investigate that the second degree homogeneous equation represents a pair of straight lines through the origin and to find the acute angle in between them.

6.1 Division of a Line Segment

The role of plane analytic geometry is so vital to the study of **"relationships between an equation and a graph."** Plane analytic geometry is that branch of geometry that ties together the geometric concept of position with an algebraic representation, namely coordinates. For example, you remember from algebra that a line can be represented by an equation. Precisely what does this mean? Can we take a statement that is true for any curve, not just for lines? The answer is given in the affirmative with the following definition.

Definition 6.1.1:[Graph of an Equation]: The graph of an equation in two variables x and y is the collection of all points p(x,y) whose coordinates (x,y) satisfy the equation.

There are two frequently asked questions in plane analytic geometry:

1. If a graph (a geometrical representation) is given, then the corresponding equation can be found easily.
2. If an equation (an algebraic representation) is given, then the corresponding graph can be viewed easily.

For example, the derivation of a circle equation using distance formula. This means that if x and y are the numbers that satisfy the circle equation, then the point (x, y) will lie on the circle. Conversely, the coordinates of any point on the circle will satisfy the circle equation.

You are probably familiar with the set of **real numbers** as well as with several of its subsets, including the counting or natural numbers, the integers, the rational numbers and the irrational numbers.

The real numbers can most easily be visualized by using a **one-dimensional** coordinate system called a **real number line.**

The line (straight line) and the point are two elements of many figures constructed on a plane. **A plane** is an infinitely large flat surface. Look at figure (6.1a) and think of **line AC**, as extending in either direction without bounds. **Line segment** DE is a measurable piece **of line AC.** The notation $|DE|$ may be used to symbolize the **length** of the line segment DE. Now turn to figure (6.1b), which shows the **ray AB.** Ray AB starts at point A and moves in the direction of point B. The union of rays AB and AC forms **angle** BAC, symbolized as $\angle BAC$, depicted in a figure (6.1b). For angle $\angle BAC$, ray AB is called the **initial ray**, and ray AC the **terminal ray**. If the initial ray visually coincides with the positive x-axis and the rotation to the terminal ray is counterclockwise, the angle is in **standard position.**

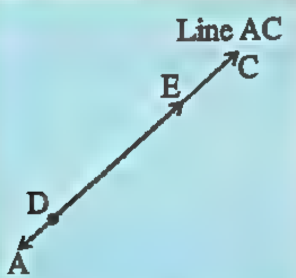


Fig. 6.1 (a)

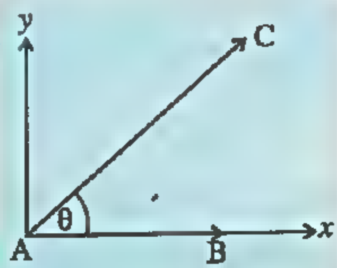


Fig. 6.1 (b)

i) Calculation of distance between two points

The study of plane analytic geometry is greatly facilitated by the use of vectors. In this unit, the vector notation is used to develop the distance formula as well as the lines in different situations. The distance between the two points can be found by the following methods:

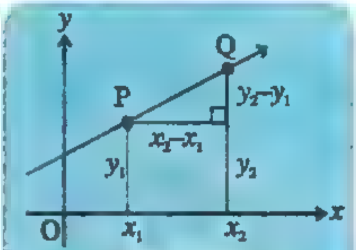


Fig. 6.2(a)

Pythagoras Theorem: If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the two points in the xy–plane, then the distance d between the given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is obtained by applying the theorem of Pythagoras to triangle PQR:

$$(PQ)^2 = (PR)^2 + (QR)^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 \qquad (1\text{-a})$$
$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d, say$$

Directed Line Segment PQ: If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the two points in the xy–plane, then the directed line segment associated to initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$ is PQ.The components of the directed line segment **PQ** are:

$$OP + PQ = OQ$$
$$PQ = OQ - OP, \ \textit{position vectors}$$
$$= (x_2, y_2) - (x_1, y_1) \qquad (1\text{-b})$$
$$= (x_2 - x_1, y_2 - y_1)$$

The magnitude or the length of the directed line segment PQ is also the distance d from point $P(x_1, y_1)$ to point $Q(x_2, y_2)$:

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d \qquad (1\text{-c})$$

In other words, squaring both sides of the directed line segment PQ to obtain:

$$(PQ)^2 = [(x_2 - x_1)i + (y_2 - y_1)j]^2$$
$$= (x_2 - x_1)^2 i.i + (y_2 - y_1)^2 j.j + 2(x_2 - x_1)(y_2 - y_1) i.j$$

$$= (x_2 - x_1)^2 i.i + (y_2 - y_1)^2 j.j + 2(x_2 - x_1)(y_2 - y_1)|i||j|\cos\theta$$

It is known that $i.i = j.j = 1,\ i.j = |i||j|\cos\theta, |i| = |j| = 1$

$$(PQ)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\theta$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos(\pi/2), \theta = \pi/2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2,\ \ \cos(\pi/2) = 0$$

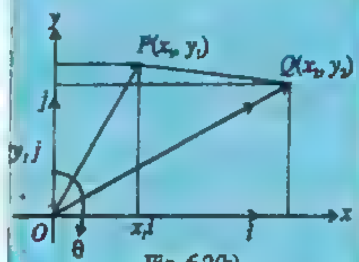


Fig. 6.2(b)

$$|PQ|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,\ (PQ)^2 = |PQ|^2$$

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d,\ say$$

This is the distance from the point $P(x_1, y_1)$ to point $Q(x_2, y_2)$ in the xy-plane.

It is important to note that

- the distance from the origin O(0,0) to point $P(x_1, y_1)$ is obtained by inserting $x_2 = y_2 = 0$ in result (1) :

$$d = |OP| = \sqrt{x_1^2 + y_1^2}$$

- the distance from the origin O(0,0) to point $Q(x_2, y_2)$ is obtained by inserting $x_1 = y_1 = 0$ in result (1):

$$d = |OQ| = \sqrt{x_2^2 + y_2^2}$$

- if the line segment PQ is horizontal, then the distance from the point $P(x_1, y_1)$ to point $Q(x_2, y_2)$ is obtained by inserting $y_1 = y_2$ in result (1):

$$d = |PQ| = \sqrt{(x_2 - x_1)^2}$$

- if the line segment PQ is vertical, then the distance from point $P(x_1, y_1)$ to point $Q(x_2, y_2)$ is obtained by inserting $x_1 = x_2$ in result (1):

$$d = |PQ| = \sqrt{(y_2 - y_1)^2}$$

Example 6.1.1:[Distance Formula]: Find the distance between the two points P(3,–2) and Q(–1, –5).

Solution: Result (1) for the following assumptions

$$P(x_1, y_1) = (3, -2),\ Q(x_2, y_2) = (-1, -5)$$

is used to obtain the distance d in between the two points P and Q:

$$d = \sqrt{x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-1-3)^2 + [(-5-(-2)]^2}$$
$$= \sqrt{(-4)^2 + (-3)^2} = \sqrt{25} = 5$$

Example 6.1.2:[Distance Formula]: Show that the triangle with vertices A (–2,–3), B(2,1) and C(–2,5) is a right angled triangle.

Solution: If A, B and C are the three vertices of a triangle ABC, then the lengths of the sides AB, BC and CA of the triangle ABC are the following:

$$|AB| = \sqrt{[2-(-2)]^2 + [1-(-3)]^2} = \sqrt{32}, \quad AB = (2+2, 1+3)$$
$$|BC| = \sqrt{(-2-2)^2 + (5-1)^2} = \sqrt{32}, \quad BC = (-2-2, 5-1)$$
$$|CA| = \sqrt{[-2-(-2)]^2 + [-3-5]^2} = \sqrt{64}, \quad CA = (-2+2, -3-5)$$

The square of the lengths of the two sides $|AB|^2$ and $|BC|^2$ of the triangle ABC equals the square of the length of its one side $|CA|^2$:

$$|AB|^2 + |BC|^2 = 32 + 32 = 64 = |CA|^2$$

Thus, the triangle ABC with vertices A, B and C is a right angled triangle with right angle at vertex B and CA as its hypotenuse. This is show in the figure (6.3):

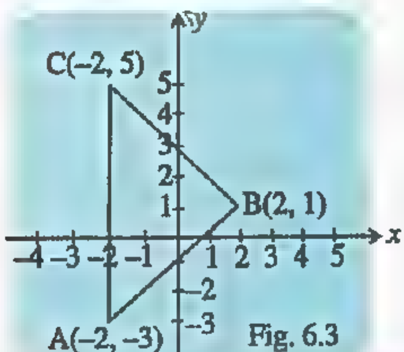


Fig. 6.3

ii) *Coordinates of a point that divides the line segment in the given ratio*

Take $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the initial and terminal points of a line segment PQ and R(x, y) is a point that divides PQ in the ratio $m_1 : m_2$. If r_1, r_2 and r are the position vectors of P, Q and R, then

$$r_1 = (x_1, y_1) = x_1 i + y_1 j, \quad r_2 = (x_2, y_2) = x_2 i + y_2 j, \quad r = (x, y) = xi + yj$$

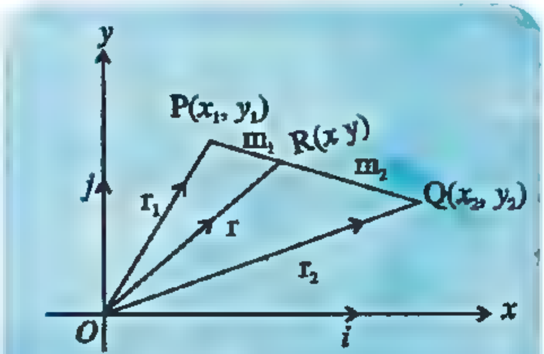


Fig. 6.4

If $\frac{PR}{RQ}=\frac{m_1}{m_2}$, then,

$$PR=\frac{m_1}{m_1+m_2}PQ$$

$$=\frac{m_1}{m_1+m_2}(OQ-OP)=\frac{m_1}{m_1+m_2}(r_2-r_1),\quad OP+PQ=OQ$$

If $OP+PR=r_1+\frac{m_1}{m_1+m_2}(r_2-r_1)$, $\quad OP+PR=OR$

then the position vector of OR is:

$$OR=OP+PR$$

$$=r_1+\frac{m_1}{m_1+m_2}(r_2-r_1)$$

$$r=\frac{m_2r_1+m_1r_2}{m_1+m_2},\qquad OR=r$$

$$(x,y)=\frac{m_2(x_1,y_1)+m_1(x_2,y_2)}{m_1+m_2},\quad \textit{components form}$$

Equating x and y components to obtain the coordinates of R(x,y)

$$(x,y)=\left(\frac{m_1x_2+m_2x_1}{m_1+m_2},\frac{m_1y_2+m_2y_1}{m_1+m_2}\right)\qquad (2)$$

that divides the line segment PQ in the ratio $m_1:m_2$.

It is important to note that

- if R is the midpoint of the line segment PQ, then, $m_1 = m_2$ and the coordinates of the midpoint R of the line segment PQ are:

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \qquad (2a)$$

- The coordinates of the point that divides the line segment PQ joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ externally in the ratio $m_1 : m_2$ (m_1 or m_2 is negative) are:

$$(x, y) = \left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y}{m_1 - m_2}\right). \qquad (2b)$$

Example 6.1.3:[Ratio $m_1 : m_2$]: Find the coordinates of the point which divides the line segment PQ joining the two points

a. P(1, 2) and Q(3, 4) in the ratio 5 : 7.

b. P(3, 4) and Q(–6, 2) in the ratio 3: –2.

Solution:

a. If R(x, y) is a point that divides the line segment PQ in the ratio 5:7, then the coordinates of R(x, y) is obtained through result (2):

$$(x, y) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right), m_1 = 5, m_2 = 7, P(1,2), Q(3,4)$$

$$= \left(\frac{5(3)+7(1)}{5+7}, \frac{5(4)+7(2)}{5+7}\right) = \left(\frac{11}{6}, \frac{17}{6}\right).$$

b. If R(x, y) is a point that divides the segment PQ in the ratio 3:–2, then the coordinates of R(x, y) is obtained through result (2b):

$$(x, y) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

$$= \left(\frac{3(-6)+(-2)(3)}{3-2}, \frac{3(2)+(-2)(4)}{3-2}\right) = (-24, -2), \quad m_1 = 3, m_2 = -2$$

iii) Concurrency of the medians and angle bisectors of a triangle

- *For the medians of a triangle are concurrent, the procedure developed is as under:*

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle ABC and P, Q and R are the midpoints of the sides AB, BC and CA, then the coordinates of the midpoint Q through result (2a) are:

$$Q\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right), m_1 = m_2$$

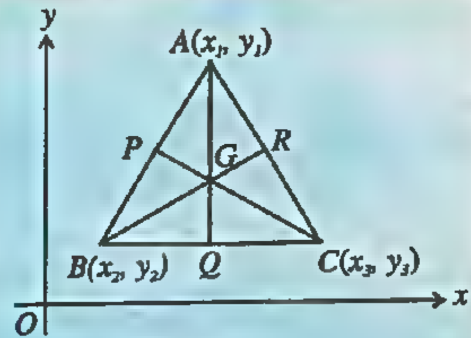


Fig. 6.5

If G(x, y) is the centroid of the triangle ABC (centroid of a triangle ABC is a point that divides each median in the ratio 2:1), then, the coordinates of the point G that divides the median AQ in the ratio $m_1 : m_2 = 2:1$ are:

$$G(x, y) = \left(\frac{2\left(\frac{x_2+x_3}{2}\right)+x_1}{2+1}, \frac{2\left(\frac{y_2+y_3}{2}\right)+y_1}{2+1}\right) \quad (3)$$

$$= \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Similarly, the coordinates of the point G(x, y) that divides the medians BR and CP each in the ratio 2: 1 are respectively:

$$G(x, y) = \left(\frac{2\left(\frac{x_1+x_3}{2}\right)+x_2}{2+1}, \frac{2\left(\frac{y_1+y_3}{2}\right)+y_2}{2+1}\right) \quad (4)$$

$$= \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

$$G(x,y)=\left(\frac{2\left(\frac{x_1+x_2}{3}\right)+x_3}{2+1},\frac{2\left(\frac{y_1+y_3}{2}\right)+y_2}{2+1}\right) \quad (5)$$

$$=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$$

Thus, the point G(x, y) lies on each median and consequently the medians of the triangle ABC are concurrent.

Example 6.1.4: [Median Bisector]: Find the centroid of the triangle ABC, whose vertices are A (3,–5), B (–7, 4) and C(10, –2).

Solution: If A(3, –5), B (–7, 4) and C (10, –2) are the vertices of the triangle ABC, and P, Q and R are the midpoints of the sides AB, BC and CA, then, the coordinates of the midpoint Q through result (2a) are:

$$Q=\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right)=\left(\frac{-7+10}{2},\frac{4-2}{2}\right)=\left(\frac{3}{2},1\right)$$

If G(x, y) is the centroid of the triangle ABC that divides each median in the ratio 2:1, then, the coordinates of the point G(x, y) (that divides the median AQ in the ratio $m_1 : m_2 = 2 : 1$) through result (3) are:

$$G(x,y)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)=\left(\frac{3-7+10}{3},\frac{-5+4-2}{3}\right)=(2,-1)$$

Similarly, the coordinates of the centroid G(x, y) that divides the medians BR and CP each in the ratio 2:1 are of course (2,–1).

- ***For the angle bisectors of a triangle are concurrent, the procedure developed is as under:***

If ABC is a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, whose lengths are $|AB| = c$, $|BC| = a$ and $|CA| = b$, then, the position vectors of A, B and C are respectively:

$$r_1=(x_1,y_1)=x_1i+y_1j,\quad r_2=(x_2,y_2)=x_2i+y_2j,\quad r_3=(x_3,y_3)=x_3i+y_3j$$

Consider AD, BE and CF are the internal bisectors of the angles A, B and C that meet at incentre G. This is shown in Fig. 6.6

If AD is the internal bisector of angle A, then:

$$\frac{BD}{DC}=\frac{BA}{AC} \quad or \quad \frac{BD}{DC}=\frac{c}{b} \Rightarrow BD:DC=c:b \qquad (6)$$

This means that D divides BC internally in the ratio c:b and the position vector of D is therefore:

$$\frac{cr_3+br_2}{c+b}$$

Again,

$$\frac{BD}{c}=\frac{DC}{b}=\frac{BD+DC}{c+b}$$
$$=\frac{a}{c+b} \qquad (7)$$
$$BD=\frac{ac}{c+b}$$

If BG is the internal bisector of the angle B, then,

$$\frac{DG}{AG}=\frac{BD}{AB}=\frac{\frac{ac}{b+c}}{c}=\frac{a}{b+c} \Rightarrow DG;GA=a:(b+c)$$

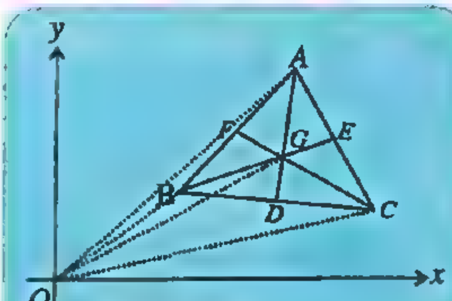


Fig. 6.6

The position vector of G(*x*, y) is:

$$r=\frac{ar_1+(b+c).\left(\frac{cr_3+br_2}{b+c}\right)}{a+b+c} \qquad (8)$$
$$=\frac{ar_1+br_2+cr_3}{a+b+c}=\frac{a(x_1,y_1)+b(x_2,y_2)+c(x_3,y_3)}{a+b+c}$$

The coordinates of the centroid G(*x*, y) is obtained from equation (8) by equating the *x* and y components:

$$G(x, y)=\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right) \qquad (9)$$

Similarly, the internal bisector of the angle C also passes through the point G(x, y). Thus, the angle bisectors of a triangle ABC are concurrent and G(x, y) is the point of concurrency.

Exercise 6.1

1. In each case, find the length of the line segment PQ joining the two points:

   a. P(–3,–5), Q(–3,–7) b. P(–4,3), Q(7,2)

   c. P(5,5), Q(5,–6) d. P(0,0), Q(0,3)

2. The three points are A(–1,3), B(2,1) and C(5,–1). Show that $|AB|+|BC|=|AC|$.

3. Show that the points A(0,–2) B(3,1), C(0,4) and D(–3,1) are the vertices of a square ABCD.

4. Show that the points A (13,-1), B(-9,3) and C (-3 -9) are the vertices of a right angled triangle ABC.

5. Show that the points A(21,–2), B(15,10), C(–5,0) and D(1,–12) are the vertices of a rectangle ABCD.

6. Show that the points A(4,3), B(3,1) and C(1,2) are the vertices of an isosceles triangle ABC.

7. Find the point which is equidistant from the points A(2,1), B(–4,3) and C(–6, 5).

8. In each case, find the midpoint of the line segment PQ joining the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$:

   a. P(10,20), Q(–12,–8) b. P(a,–b), Q(–a,b)

   c. P(a–b,a+b), Q(a+b, a–b) d. P(–1/2,–2/4), Q(3/5,4/7)

9. In each case, find the coordinates of the point R(x,y) which divides the line segment PQ joining the two points

   a. P(1,2), Q(3,4) in the ratio 5:7.

   b. P(3,4), Q(–6,2) in the ratio 3:–2.

   c. P(1,–2) Q(4,7) in the ratio –2:3.

   d. P(–6,7), Q(5,–4) in the ratio 2/7:1.

10. In each case, find the coordinates of the other end point (P or Q) of the line segment PQ, when the midpoint and one end point of the line segment PQ are the following:

a. D(2,–1), P(–1,–3), Q ? b. D(–3,–4), Q(2,4), P ?
c. D(1,2), P(4,5), Q ? d. D(–3,5), Q(3,2), P ?

11. In each case, in what ratio is the line segment PQ (joining the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$) divided by the point R(x, y):

a. P(8,10), Q(–12,6), R(–4/7,58/7) ?
b. P(4,4), Q(5,-2), R(-1,2) ?
c. P(4,4), Q(10,16), R(6,8) ?
d. P(-1,-2),Q(3,2), R(2,1)?

12. Find the centroid of the triangle ABC, whose vertices are the following:

a. A(4,–2), B(–2,4), C(5,5)
b. A(3,5), B(4,6), C(3,5)
c. A(1,1), B(–2,–2), C(4,5)
d. A(1,4), B(2,6), C(3,–1)

6.2 Slope of a Straight Line

The slope of a line is a measure of the **"steepness"** of the line, and whether it rises (goes up), or falls (goes down) when moving from left to right. The line from A to B rises up, while the line from C to D goes down are depicted in the figure (6.7):

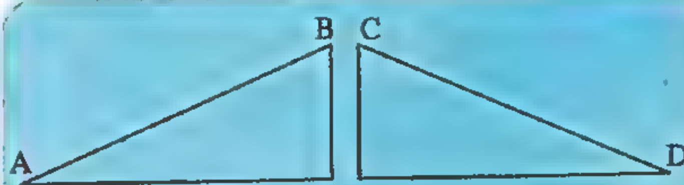


Fig. 6.7

i) Definition of the slope of a line

Definition:6.2.1[Slope of a Line]: The graph of a line can be drawn knowing only one point on the line if the "steepness" of the line is known, too. The number that represents the "steepness" of a line is called the **slope** of that line.

If move off the line horizontally to the right first or move up or down (vertically) to return to the line, then the slope of the line is the "steepness" defined as the ratio of the vertical rise to the horizontal run:

$slope = \frac{rise}{run}$, the run is always a movement to the right

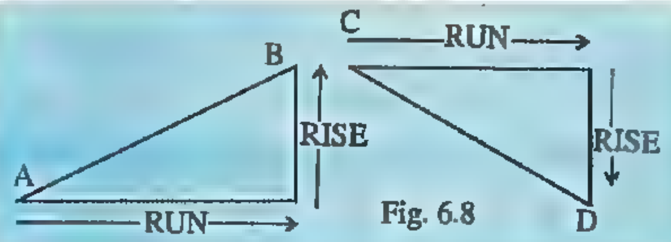


Fig. 6.8

ii) Derivation of slope formula

Mathematically, if any two points on a line are available, then their join makes a constant angle with a fixed direction and the angle so formed is independent of the choice of the two points on the line. This is a precise way of saying that any line has a constant slope. It is customary to measure the angle θ which a line makes with the positive direction of the x-axis. The quantity $\tan\theta$ is defined to be the slope of the line and is denoted by **m**. The slope of a line is also referred to **gradient** of the line.

For illustration, if $A(x_1, y_1)$ and $B(x_2, y_2)$, where $x_1 \neq x_2$, are any two points, then their join develops a line L that makes a constant angle θ with the x-axis. Draw AM, and BN parallel to y-axis and AL parallel to x-axis.

The slope **m** of a line L through the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is therefore:

$$\begin{aligned} m = \tan\theta &= \frac{LB}{AL} \\ &= \frac{NB - NL}{MN} \qquad\qquad (10) \\ &= \frac{NB - AM}{ON - OM} = \frac{y_2 - y_1}{x_2 - x_1} \end{aligned}$$

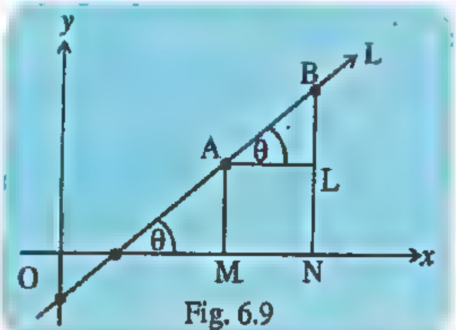


Fig. 6.9

Example 6.2.1:[Slope]: Find the slope m of the line L through the two points

a. E(2,4) and F(4,6)   b. M(3,1) and N(–1,3)

Solution:

a. The given two points E(2,4) and F(4,6) is forming a line L, whose slope is:

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{6-4}{4-2}=\frac{2}{2}=1$$

b. The given two points M(3,1) and N(–1,3) is forming a line L, whose slope is:

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{3-1}{-1-3}=\frac{2}{-4}=-\frac{1}{2}$$

iii) The condition at which the two straight lines with given slopes are

- ***Parallel to each other***

If L_1 and L_2 are the two lines having slopes m_1 and m_2, then the lines L_1 and L_2 are **parallel** if they make the same angle with the x-axis, that means they have the same slope. Conversely, if two lines L_1 and L_2 have the same slope, then they will make the same angle with the x-axis and the lines L_1 and L_2 are therefore parallel for which :

$$m_1 = m_2 \qquad (11)$$

It is important to note that the lines parallel to x-axis have **zero slopes** whereas the lines parallel to y-axis have the **slope** ∞.

- ***Perpendicular to each other***

If L_1 and L_2 are the two perpendicular lines make the angles α and β with the x-axis, then the slopes of the lines L_1 and L_2 are respectively $m_1 = \tan\alpha$ and $m_2 = \tan\beta$. From the Fig. 6.10, it is clear that

$$\frac{\pi}{2} = \beta - \alpha$$

$$\beta = \frac{\pi}{2} + \alpha$$

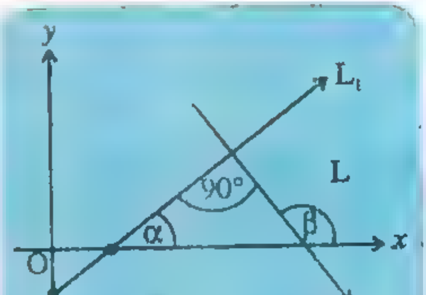


Fig. 6.10

Taking tan on both sides to obtain:

$$\tan\beta = \tan\left(\frac{\pi}{2} + \alpha\right)$$

$$= -\cot\alpha = -\frac{1}{\tan\alpha} \qquad (12)$$

The given lines L_1 and L_2 are found perpendicular, since the product of their slopes equals -1:

$$m_1 m_2 = \tan\alpha \tan\beta = \tan\alpha\left(-\frac{1}{\tan\alpha}\right) = -1 \qquad (13)$$

6.3 Equation of a Straight Line Parallel to Co-ordinate Axes

i) *The equation of a straight line parallel to*

- ***x-axis and at a distance a from it***
- ***y-axis and at a distance b from it***

Let AB be a straight line parallel to x–axis that intersect the y–axis at a point N(0, a). Take any point P(x, y) on a line AB. The position vectors OP and ON of points P(x, y) and N(0, a) are respectively :

r=OP = (x, y) = xi + yj, ON=(0, a) = 0+aj

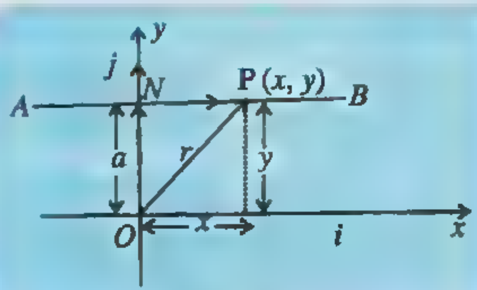


Fig. 6.11

From the Fig. 6.11

$$OP = ON + NP$$

$$NP = OP - ON$$

$$= (xi + yj) - aj = xi + (y-a)j = (x, y-a)$$

If $NP \perp ON$, then the dot product between these two vectors NP and ON is zero that gives the equation of a line parallel to the x–axis

$$ON.NP = 0$$

$$(0,a).(x, y-a) = 0$$

$$0 + a(y-a) = 0$$

$$ay - a^2 = 0$$

$$ay = a^2$$

$$y = a \qquad (14)$$

and is at a distance **a** units from it. Result (14) is true for all points on a line AB.

It is important to note that

- if a=0, then the straight line coincides with the x–axis and its equation becomes y=0.
- if **a** is positive, then the straight line lies above the x–axis, otherwise the line lies below the x–axis, in case **a** is negative. The vector equation of the x–axis is r.j=0.
- the equation of a straight line parallel to the y–axis is y=b, which is at a distance **b** from it. The equation of the y–axis is $x = 0$ and the vector equation of y–axis is r.i = 0.

6.4 Standard Form of Equation of a Straight Line

Because of its simplicity, linear equation (line) is used in many applications to describe relationships between two variables. We shall see some of these applications in this unit. First, we need to develop some standard forms that are related to linear equations.

i) *Intercepts of a line and the derivation of the equation of a line in different forms*

Definition 6.4.1: [Intercepts]: If a straight line AB intersects x–axis at C and y–axis at D, then OC is called the **x-intercept** of AB on the x–axis and OD is called the **y-intercept** of AB on the y–axis.

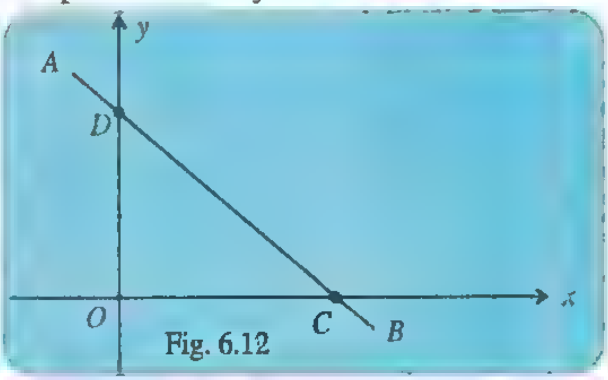


Fig. 6.12

Example 6.4.1:[Intercepts]: Find the x and y intercepts of a line 2x+4y+6=0.

Solution: The x-intercept of a line is obtained by putting y=0 in a line:

$$2x+4y+6=0$$

$$2x+4(0)=-6 \Rightarrow 2x=-6 \Rightarrow x=-3$$

The y-intercept of a line is obtained by putting x=0 in a line:

$$2x+4y+6=0$$

$$2(0)+4y=-6 \Rightarrow 4y=-6 \Rightarrow y=-3/2$$

In general a line in two dimensional spaces can be determined by specifying its **slope and just one point**. In different situations, the lines developed are the following:

- ***Slope-Intercept Form***

Let L be the line develops the y-intercept c on the y-axis. The line L also makes an angle θ with the positive direction of the x-axis that develops a slope $m = \tan\theta$.

Let P(x,y) be any point on the line L. Draw PM parallel to y-axis and CN parallel to x-axis. This gives:

$$CN = OM = x, \quad NP = MP - MN = MP - OC = y - c$$

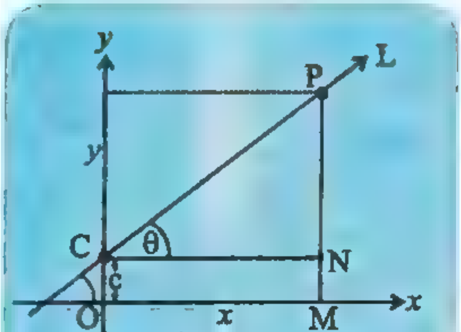


Fig. 6.13

In triangle PCN, the angle is $\angle PNC = 90^0$ and the slope of the line L is giving the slope-intercept form of the line L:

$$\frac{NP}{CN} = \tan\theta$$

$$\frac{y-c}{x} = \tan\theta \Rightarrow y - c = x\tan\theta \Rightarrow y = x\tan\theta + c = mx + c \qquad (15)$$

In y = mx + c, m is the slope and c is the y-intercept of the line L on the axis of y. If the straight line L passes through the origin (0, 0), then c=0 and the equation of line (15) becomes y = mx.

Example 6.4.2:[Slope–Intercept Form]: Find an equation of the line with slope 4, when the y–intercept is 6.

Solution: Result (15) is used for the assumptions m=4, c=6 to obtain the required slope–intercept form of a line:

$y=4x+6$

• *Point–Slope Form*

If L is a line passing through the point $A(x_1, y_1)$ and P(x, y) is any point on a line L, then the slope of the line L develops the point-slope form of a line L through the point $A(x_1, y_1)$:

$$m=\frac{y-y_1}{x-x_1} \qquad (16)$$

$$y-y_1=m(x-x)$$

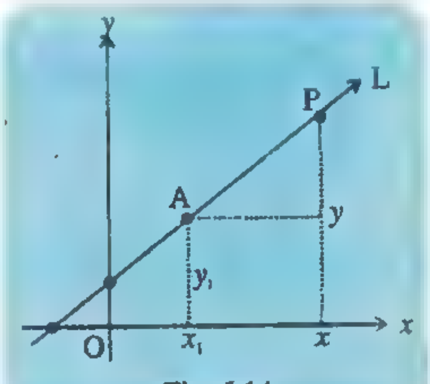


Fig. 6.14

Example 6.4.3:[Point–Slope Form]: Find an equation of a line with slope 4 and passes through the point (2,4).

Solution: Result (16) is used for the assumptions $m=4,\ A(x_1,y_1)=A(2,4)$ to obtain the required point–slope form of a line:

$$y-4=4(x-2)$$

$$-4x+y-4+8=0 \Rightarrow -4x+y+4=0 \Rightarrow 4x-y-4=0$$

- ***Two–Point Form***

If L is a line passing through the two points $A(x_1,y_1)$ and $B(x_1,y_1)$, then the slope of the line L is:

$$m=\frac{y_2-y_1}{x_2-x_1} \qquad (17)$$

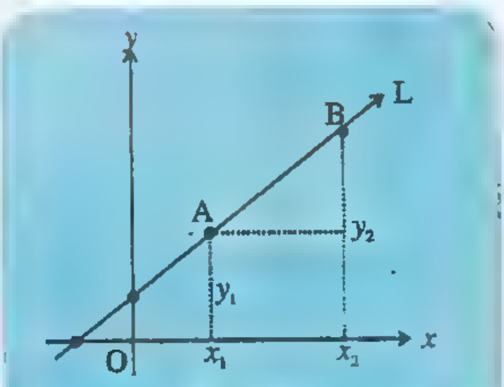


Fig. 6.15

The point-slope form of a line L through the $A(x_1, y_1)$ with slope m is

$$y - y_1 = m(x - x_1) \qquad (18)$$

Equation (17) is used in Equation (18) to obtain the equation of the **two-point form** of a line L through the two points $A(x_1, y_1)$ and $B(x_2, y_2)$:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\frac{(x - x_1)}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \qquad (19)$$

Example 6.4.4:[Two-Point Form]: Find an equation of a line that passes through the two points P(–1,–2) and Q(–5,0).

Solution: Result (19) is used for the assumptions $P(x_1, y_1) = P(-1,-2)$, $Q(x_2, y_2) = Q(-5,0)$ to obtain the required two–point form of a line:

$$y-(-2) = \frac{0-(-2)}{-5-(-1)}[x-(-1)]$$

$$y+2 = \frac{2}{-4}(x+1)$$

$$-4y-8 = 2x+2 \Rightarrow 2x+4y+10=0 \Rightarrow x+2y+5=0$$

- ***Double–Intercepts Form***

If a line L intersects the x-axis and y-axis at points A and B, then OA=a and OB=b are the x and y-intercepts of the line L.

Let P(x, y) be any point on the line L. Draw PM parallel to y-axis and PN parallel to x-axis. From the figure (6.16), the comparison of similar triangles ΔBNP and ΔPMA is giving the equation of double-intercept form of a line L:

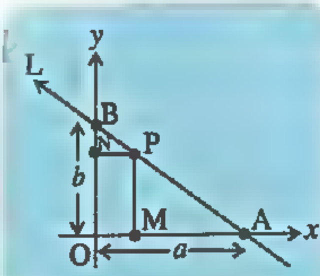


Fig. 6.16

$$\frac{NB}{MP} = \frac{NP}{MA}$$

$$\frac{OB - ON}{ON} = \frac{OM}{OA - OM}$$

$$\frac{b-y}{y} = \frac{x}{a-x}$$

$$bx + ay = ab$$

$$\frac{bx}{ab} + \frac{ay}{ab} = 1$$

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (20)$$

Example 6.4.5:[Double–Intercepts Form]: Find the equation of a line whose x and y intercepts are (3,0) and (0,4) respectively.

Solution: Result (22) is used for the assumptions $a=3$, $b=4$ to obtain the required line:

$$\frac{x}{3}+\frac{y}{4}=1$$

$$\frac{4x+3y}{12}=1 \Rightarrow 4x+3y=12 \Rightarrow 4x+3y-12=0$$

- ***Symmetric Form***

Let a line L through point $A(x_1, y_1)$ makes an angle θ with the positive direction of the x-axis.

If P(x, y) is any point on the line L, then AP= r. If we allow r to vary with any positive or negative values, then P will take any position on the line L. Conversely, if P is given to be any point on the line L, then the unique value of r can be found which in fact is the distance of P from A. Thus, it follows that r serves as a parameter of point P.

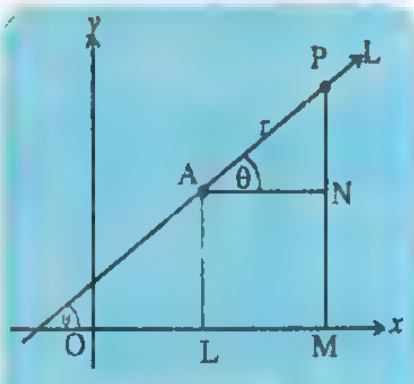


Fig. 6.17

To find the coordinates of a point P in terms of the parameter r, let us draw AL and PM parallel to y-axis and AN parallel to x-axis, that with the following assumptions

$$\begin{aligned} OM &= OL+LM = OL+AN \\ MP &= MN+NP = LA+NP \end{aligned} \quad (21)$$

dvelops the parametric equations of a line L through the point $A(x_1, y_1)$ at an angle θ:

$$\begin{aligned} OM &= OL+LM & MP &= MN+NP \\ &= OL+AN & &= LA+NP \end{aligned} \quad (22)$$

$$x = x_1 + r\cos\theta,\ \cos\theta = AN/r \qquad y = y_1 + r\sin\theta,\ \sin\theta = NP/r$$

The parametric equations (22) automatically give the symmetric form of a line L after simplification:

$$\left.\begin{array}{l}\dfrac{x-x_1}{\cos\theta}=r\\ \dfrac{y-y_1}{\sin\theta}=r\end{array}\right\} \Rightarrow \frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \qquad (23)$$

• Normal Form

A normal to a line is a line segment drawn from a point perpendicular to the given line. This is shown in the Fig. 6.18a. The normal form of a line is the equation of a line in terms of the length of the perpendicular on it from the origin and that perpendicular makes an angle with the x-axis.

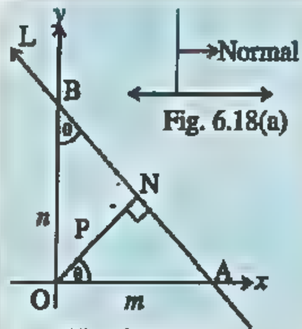


Fig. 6.18(a)
Fig. 6.18(b)

If a line L intersects the x-axis and y-axis at points A and B, then m and n are the x and y-intercepts of the line L. Draw ON perpendicular to line L that provides the perpendicular distance p from the origin on the line L which is denoted by ON= p.

If ON makes an angle θ with the positive direction of the x-axis, then the x and y-intercepts of the line L are respectively:

$$\cos\theta=\frac{p}{m}\Rightarrow m=p\sec\theta$$

$$\sin\theta=\frac{p}{n}\Rightarrow n=p\,co\sec\theta \qquad (24)$$

If m and n are the x and y-intercepts of a line L, then the intercept form of a line L develops

$$\frac{x}{m}+\frac{y}{n}=1$$

$$\frac{x}{p\sec\theta}+\frac{y}{p\cos ec\theta}=1$$

$$x\cos\theta+y\sin\theta=p \qquad (25)$$

the **normal form equation** of the given line making an angle θ with the positive direction of the x-axis and whose perpendicular distance from the origin is p.

Example 6.4.6:[Normal Form]: Find the corresponding equation of a line, if the length of the perpendicular distance from the origin on a line is 3 units that makes an angle of 120°.

Solution: Result (25) is used for the assumptions $p = 3$, $\theta = 120^0$ to obtain the required equation of a line:

$$x\cos 120^0 + y\sin 120^0 = 3$$

$$\frac{-1}{2}x + \frac{\sqrt{3}}{2}y = 3 \Rightarrow -x + \sqrt{3}y = 6 \Rightarrow x - \sqrt{3}y + 6 = 0$$

ii) *A linear equation in two variables is a straight line*

A first degree polynomial $p_1(x) = a_1x + a_0$ is rearranged to obtain an equation of the form

$$-a_1x + p_1(x) - a_0 = 0 \quad \text{with } -a_1 = a, p_1(x) = y, -a_0 = b$$
$$ax + by + c = 0$$

is then called the **general equation of the straight line**. Here a, b and c are constants while x'and y are variables. Remember, The first degree polynomial $p_1(x)$ is also called the linear algebraic equation and is denoted by $p_1(x) = f(x)$.

For logical proof, if $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ are the three points on the locus represented by the straight line

$$ax + by + c = 0 \qquad (26)$$

then,

$P(x_1, y_1)$ on the line (26) gives:	$ax_1 + by_1 + c = 0$	(27)
$Q(x_2, y_2)$ on the line (26) gives:	$ax_2 + by_2 + c = 0$	(28)
$R(x_3, y_3)$ on the line (26) gives:	$ax_3 + by_3 + c = 0$	(29)

The three lines from equation (27) to equation (29) develops a homogeneous system of three linear equations in three unknowns a, b and c:

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow Ax = 0 \qquad (30)$$

The homogeneous system of linear equations (30) defines a nontrivial solution only if the determinant of a coefficient matrix A of the system (30) is zero:

$$\det(A)=0$$

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0 \qquad (31)$$

$$x_1(y_2-y_3)-y_1(x_2-x_3)+(x_2y_3-x_3y_2)=0$$

Equation (31) is rearranged to obtain:

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0 \qquad (32)$$

Multiply both sides of equation (32) by $\frac{1}{2}$ to obtain the area of the triangle formed by P, Q and R that equals zero:

$$\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]=0 \qquad (33)$$

$$\textit{Area of the triangle } PQR=0$$

Since the three points P, Q and R lying on the locus (26) are collinear. Hence the locus (26) represents a straight line.

iii) General form

of a straight line is reducible to other standard forms

Any standard form of a line can also be determined from the general form of a line (26).

- To reduce the general form (26) to the point–slope form of a line, we need to involve the following steps:

$$ax+by+c=0$$
$$by=-ax-c$$
$$y=-\frac{a}{b}x-\frac{c}{b}$$
$$=mx+c_1,\quad slope=m=-a/b,\quad y\text{-intercept}=c_1=-c/b$$

- To reduce the general form (26) to the double–intercept form, we need to involve the following steps:

$$ax+by+c=0$$
$$ax+by=-c$$
$$\frac{ax}{-c}+\frac{by}{-c}=1, \quad \textit{divide out by} -c$$
$$\frac{x}{-\frac{c}{a}}+\frac{y}{-\frac{c}{b}}=1$$
$$\frac{x}{a_1}+\frac{y}{b_1}=1, \quad x\text{-}intercept=a_1=-c/a,\ y\text{-}intercept=b_1=-c/b$$

- To reduce the general form (26) to normal form , we need to involve the following steps:
  From the Fig. 6.18b, the angles along the positive directions of the x and y-axis are the following:
  $$\cos\theta=\frac{p}{m},\quad \sin\theta=\frac{p}{n}$$
  The values of $\cos\theta$ and $\sin\theta$ are used in the trigonometric identity $\cos^2\theta+\sin^2\theta=1$ to obtain p:
  $$\cos^2\theta+\sin^2\theta=1$$

$$\frac{p^2}{m^2}+\frac{p^2}{n^2}=1$$
$$p^2\left(\frac{1}{m^2}+\frac{1}{n^2}\right)=1 \qquad (34)$$
$$p^2\left(\frac{m^2+n^2}{m^2n^2}\right)=1$$
$$p^2=\frac{m^2n^2}{m^2+n^2} \Rightarrow p=\pm\frac{mn}{\sqrt{m^2+n^2}}$$

This p is the perpendicular distance from the origin to the line $\frac{x}{m}+\frac{y}{n}=1$ (i.e. nx+my-mn=0). Of course, the perpendicular distance from the origin to the line $ax+by+c=0$ must be:

$$p=\frac{|c|}{\sqrt{a^2+b^2}} \qquad (35)$$

For converting the general form (26) to normal form, divide the line $ax+by+c=0$ by $\frac{1}{\sqrt{a^2+b^2}}$ to obtain the conversion of the general form (26) in the normal form:

$$\frac{ax}{\sqrt{a^2+b^2}}+\frac{by}{\sqrt{a^2+b^2}}+\frac{c}{\sqrt{a^2+b^2}}=0 \qquad (36)$$

Example 6.4.7: [Direction Cosines]: What are the direction cosines of a line perpendicular to a line $x-5y+3=0$?

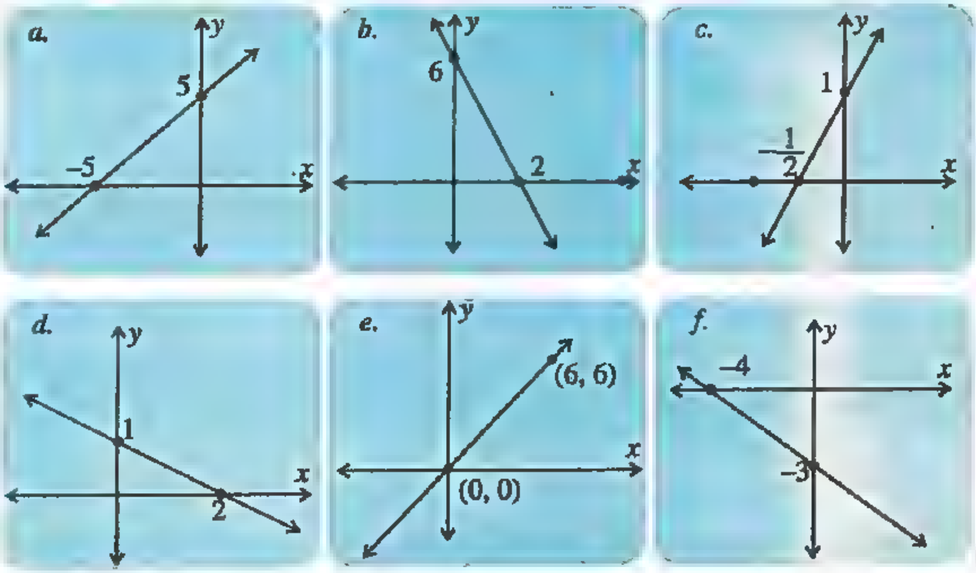
Solution: The normal form of the given line is:

$$\frac{-x}{\sqrt{26}}+\frac{5y}{\sqrt{26}}-\frac{3}{\sqrt{26}}=0$$

The direction cosines of a normal line are given by the coefficients of x and y and that are $\lambda=-\frac{1}{\sqrt{26}}$ and $\mu=\frac{5}{\sqrt{26}}$.

Exercise 6.2

1. Find the equation of lines that are represented on the coordinate planes:



2. Arrange these lines in order of steepness (shallowest to steepest):

   a. $y=\frac{2}{5}x+4$ b. $y=\frac{1}{3}x-5$ c. $y=x-3$

   d. $y=0.3x+6$ e. $y=0.01x-200$

3. In each case, find the slope, if it is defined:

   a. (–5, 4) and (3,6) b. (2,–1) and (–5,6)

   c. Through the origin and (–4, 6) d. $4x+7y=1$

   e. Parallel to $2y-4x=7$ f. Perpendicular to $6x=y-3$

4. What are the x– and y–intercepts for each of the following lines?

a. $y=2x+6$ b. $y=-3x+9$ c. $y=x+2$

d. $y=2x-8$ e. $y=5x+10$ f. $y=5x-10$

5. In each case, use the normal form to find out the equation of a line for the following assumptions:

a. $p=4,\ \theta=60^0$ b. $p=8,\ \theta=90^0$

c. $p=5,\ \theta=30^0$ d. $p=2,\ \theta=45^0$

6. In each case, find the equation of a line, if the x–intercept and y–intercept of the line are the following:

a. x: (4,0), y: (0,6) b. x: (–4,0), y: (0,–8)

c. x: (5,0), y: (0,10) d. x: (–1,0), y: (0,5)

7. In each case, find the equation of a line that passes through the pair of points:

a. O(0.0) and A(2,6) b. E(1,0) and F(2,5)

c. I(1,1) and J(3,3) d. M(3,1) and N(–1,3)

8. In each case, find the equation of a line that passes through the point $A(x_1, y_1)$ having slope m:

a. A(1,2), m=4 b. A(–1,–2), m=–1/2

c. A(–3,5), m=–3 d. A(7,–8), m=5

9. In each case, find the equation of a line that exists the y–intercept c and slope m:

a. c=2, m=2 b. c=4, m=8

c. c=–4, m=1/2 d. c=1/2, m–3

6.5 Distance of a Point from a Line

To obtain the distance of a point from a line, we need to develop the concepts in subsections below.

i) *Position of a point with respect to a line*

To show that the point $P(x_1, y_1)$ is on one side or on the other side of the straight line ax+by+c=0 according as the expression

$ax_1+by_1+c<0$ or $ax_1+by_1+c>0$,

the procedure developed is as under:

Let AB be the straight line $ax+by+c=0$ and $P(x_1, y_1)$ is a point above the line AB (figure (6.19a) and $P(x_1, y_1)$ is also a point below the line AB (figure (6.19b). From P draw perpendicular PM on the x–axis that cuts the line AB at a point Q whose coordinates are $Q(x_1, y_2)$.

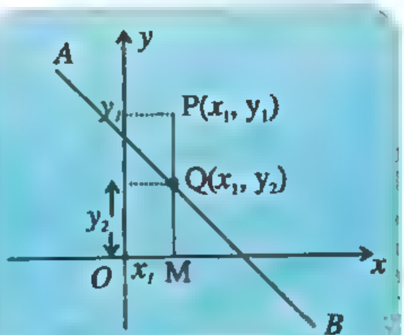


Fig. 6.19 (a)

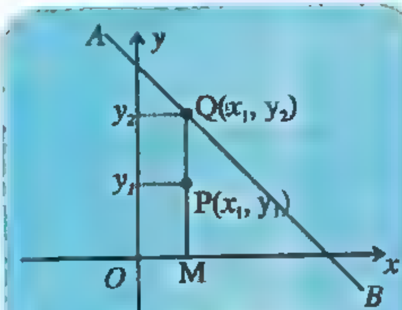


Fig. 6.19 (b)

If $Q(x_1, y_2)$ lies on the line AB, then it give:

$$ax_1 + by_2 + c = 0, \quad Q(x_1, y_2) \text{ lies on } AB$$
$$by_2 = -(ax_1 + c) \qquad (37)$$
$$y_2 = -\left(\frac{ax_1 + c}{b}\right)$$

If P lies above AB as in figure (6.19a), then:

$$MP - MQ > 0$$
$$y_1 - y_2 > 0$$
$$y_1 + \frac{ax_1 + c}{b} > 0$$
$$\frac{ax_1 + by_1 + c}{b} > 0$$
$$ax_1 + by_1 + c > 0, \quad b > 0$$

If P lies below AB as in figure (6.19b), then:

$$MP - MQ < 0$$
$$y_1 - y_2 < 0$$
$$y_1 + \frac{ax_1 + c}{b} < 0$$
$$\frac{ax_1 + by_1 + c}{b} < 0$$
$$ax_1 + by_1 + c < 0, \quad b > 0$$

Hence P lies on one side or on the other side of the line ax+by+c=0 according as $ax_1 + by_1 + c > 0$ or $ax_1 + by_1 + c < 0$.

This develops the following concepts:

- Since $ax_1 + by_1 + c > 0$ for all positions of the point $P(x_1, y_1)$ on one side of the line and $ax_1 + by_1 + c < 0$ for all positions of the point on the other side of the line. It follows that the sign of the expression changes as any point $P(x_1, y_1)$ crosses the line ax+by+c=0.
- The line ax+by+c=0 cuts the plane into two regions, one positive region (above the line), the coordinates of any point $P(x_1, y_1)$ which when substituted in ax+by+c=0, make $ax_1 + by_1 + c > 0$, and the other negative region (below the line), the coordinates of any point $P(x_1, y_1)$ which when substituted in ax+by+c=0 make $ax_1 + by_1 + c < 0$.
- **Origin case:** If the coordinates of the origin O(0,0) are substituted in the expression ax+by+c=0, the expression reduces to c=0. The origin is in positive region if $c > 0$ and in the negative region if $c < 0$.
- Two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie in the same or opposite sides of the straight line ax+by+c=0 according as the expression $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same or opposite signs.
- The point $P(x_1, y_1)$ and the origin are on the same or opposite side of the straight ax+by+c=0 according as $ax_1 + by_1 + c$ and a(0)+b(0)+c have the same or opposite signs, i.e., according as $ax_1 + by_1 + c$ and c have the same or opposite signs.

Example 6.5.1:[Position of a Point]: Determine whether the point P(10,–6) lies above or below the line 9x+10y–3=0. Show that the point and the origin lie on the same or on the opposite sides of the given line.

Solution: The given line 9x+10y–3=0 is compared to the line ax+by+c=0 to obtain the coefficient of y b=10>0:

1. The given point P(10,–6) is substituted in the given line to obtain:

$$9(10)+10(-6)-3=90-63=27>0 \text{ (above)}$$

Thus, the point P(10,–6) lies above the given line 9x+10y–3=0

2. The given point P(10,–6) and the origin O(0,0) are substituted in the given line to obtain:

$$\left.\begin{aligned} 9(10)+10(-6)-3=90-63=27>0 \\ 9(0)+10(0)-3=-3<0 \end{aligned}\right\} \text{opposite in signs}$$

Hence, the point P(10,–6)and the origin O(0,0) lie on the opposite side of the given line 9x+10y–3=0

Example 6.5.2:[Position of a Point]: Show that the points P(3,7) and Q(–3,–1) are on the same side of the line 3x–8y–7=0

Solution: The given points P(3,7) and Q(–3,–1) are substituted in the given line to obtain:

$$\left.\begin{aligned} 3(3)-8(7)-7=9-56-7=-54<0 \\ 3(-3)-8(-1)-7=-9+8-7=-8<0 \end{aligned}\right\} \text{same signs}$$

Thus, the point P(3,7) and Q(–3,–1) are on the same side of the given line 3x–8y–7=0.

ii) *Perpendicular distance from the point on a line*

If $Q(x_1, y_1)$ is any point on a line

$$ax+by+c=0 \qquad (38)$$

and $n=(a,b)$ is a nonzero vector perpendicular to the line (38) at a point $Q(x_1, y_1)$, then the distance D is the scalar projection of a vector QP (associated to any point $(P(x_0, y_0))$ onto n:

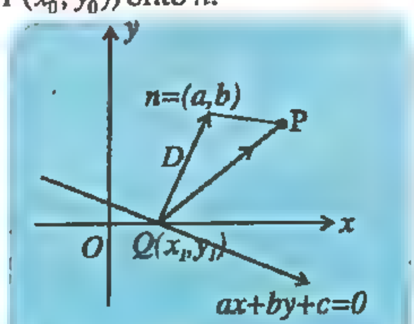


Fig. 6.20

$$D = |proj_n QP|$$

$$= \frac{|QP.n|}{|n|}$$

$$= \frac{|(x_0 - x_1, y_0 - y_1).(a,b)|}{\sqrt{a^2+b^2}}, \; n = (a,b), |n| = \sqrt{a^2+b^2}$$

$$= \frac{|a(x_0 - x_1) + b(y_0 - y_1))|}{\sqrt{a^2+b^2}}$$

$$= \frac{|ax_0 + by_0 - ax_1 - by_1|}{\sqrt{a^2+b^2}} = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}} \qquad (39)$$

Here $ax_1 + by_1 + c = 0 \Rightarrow c = -ax_1 - by_1$ is a line through $Q(x_1, y_1)$.

Example 6.5.3:[Perpendicular Distance]: Find the perpendicular distance from a point $P(2, 3)$ to a line $7x+3y-9=0$.

Solution: Result (39) is used for the assumptions $P(x_1, y_1) = P(2,3)$, $c=-9$, $a=7$, $b=3$ to obtain the perpendicular distance d from the line $7x+3y-9=0$ to the point P(2,3):

$$d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \right|$$

$$= \frac{7(2)+3(3)-9}{\sqrt{7^2+3^2}} = \frac{14}{\sqrt{58}}$$

6.6 Angle between Lines

If the two lines are available, then the angle between these two lines can found as follows:

i) The Angle between two coplanar intersecting straight lines

First method: [Basic Geometry]: Let AB and CD be the two straight lines

$y = m_1x + c_1$ and $y = m_2x + c_2$

These lines are intersecting at a point P(x, y). The line AB intersects the x–axis at a point E that makes an angle θ_1, while the line CD intersects the x–axis at a point F that makes an angle θ_2. The slopes and angles developed by the lines AB and CD are respectively:

$m_1 = \tan\theta_1, \quad m_2 = \tan\theta_2$

Let θ be the angle of intersection between the lines AB and CD, P being the point of intersection of the lines AB and CD. The basic geometry rule that the angle θ between the lines AB and CD will be:

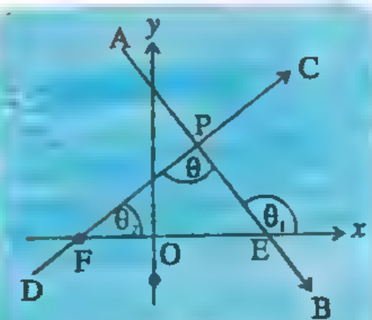


Fig. 6.21

$\theta = \theta_1 - \theta_2$

Take *tan* of both sides which leads the angle between the two lines AB and CD:

$\tan\theta = \tan(\theta_1 - \theta_2)$

$$= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$= \frac{m_1 - m_2}{1 + m_1m_2}, \quad m_1 = \tan\theta_1,\ m_2 = \tan\theta_2 \qquad (40)$$

This expression will change its sign depending upon whether we take m_1 as the slope of the first line or the second. But the change of sign is not a problem, because the positive / negative values will give supplementary angles (as $\tan(\pi - \theta) = -\tan\theta$). For example, values 1 and -1 will mean an angle of $\tan 45^0 = 1$ and $\tan 135^0 = -\tan 45^0 = -1$, both of which correspond to the same situation geometrically.

To clear this sign issue, use a modulus sign on the previous expression, which will always give a positive value (or an acute angle). The final formula looks something like this:

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| \qquad (40\text{-a})$$

Second method:[Dot Product of Unit-Vectors]: The unit vectors are the vectors lie in the same directions of the given lines. The unit vectors along the lines AB and CD are respectively $u=(\cos\theta_1, \sin\theta_1)$ and $v=(\cos\theta_2, \sin\theta_2)$.

The unit vector u of a line AB is:

$$u=(\cos\theta_1, \sin\theta_1), \cos\theta_1 = 1/\sec\theta_1$$
$$=\left(\frac{1}{\sec\theta_1}, \frac{\tan\theta_1}{\sec\theta_1}\right), |u|=1$$
$$=\left(\frac{1}{\sqrt{1+m_1^2}}, \frac{m_1}{\sqrt{1+m_1^2}}\right), \sec^2\theta_1 = 1+\tan^2\theta_1 = 1+m_1^2$$

The unit vector v of a line CD is:

$$v=(\cos\theta_2, \sin\theta_2), |v|=1$$
$$=\left(\frac{1}{\sec\theta_2}, \frac{\tan\theta_2}{\sec\theta_2}\right)$$
$$=\left(\frac{1}{\sqrt{1+m_2^2}}, \frac{m_2}{\sqrt{1+m_2^2}}\right), \quad \sec^2\theta_2 = 1+\tan^2\theta_2 = 1+m_2^2$$

The angle of intersection between the lines AB and CD is the angle of intersection in between their unit vectors u and v that can be found by taking the dot product in between the unit vectors u and v:

$$\cos\theta = \frac{u.v}{|u||v|}, \quad |u|=|v|=1$$
$$=u.v$$
$$=\left(\frac{1}{\sqrt{1+m_1^2}}, \frac{m_1}{\sqrt{1+m_1^2}}\right)\cdot\left(\frac{1}{\sqrt{1+m_2^2}}, \frac{m_2}{\sqrt{1+m_2^2}}\right)$$
$$=\frac{1}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}} + \frac{m_1m_2}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}$$
$$=\frac{1+m_1m_2}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}$$

The standard form of the angle is obtained if

$$\tan^2\theta = \sec^2\theta - 1$$

$$= \frac{1}{\cos^2\theta} - 1, \text{ use value of } \cos\theta$$

$$= \frac{(1+m_1^2)(1+m_2^2)}{(1+m_1m_2)^2} - 1$$

$$= \frac{(1+m_1^2)(1+m_2^2) - (1+m_1m_2)^2}{(1+m_1m_2)^2} = \frac{(m_1 - m_2)^2}{(1+m_1m_2)^2}$$

$$\tan\theta = \pm\sqrt{\frac{(m_1 - m_2)^2}{(1+m_1m_2)^2}}$$

$$= \pm\frac{m_1 - m_2}{1+m_1m_2} \qquad (41)$$

$$= \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$$

It is important to note that

- If $\frac{m_1 - m_2}{1+m_1m_2}$ is positive, then result (41) gives the acute angle between the lines AB and CD.
- If $\frac{m_1 - m_2}{1+m_1m_2}$ is negative, then result (41) gives the obtuse angle between the lines AB and CD.
- If one of the given lines is parallel to the y–axis, then the angle θ is not possible to obtain by formula:

  $$\tan\theta = \pm\frac{m_1 - m_2}{1+m_1m_2}$$

  Because 90^0 is the angle made by that line with the positive x–axis and $\tan 90^0 = \infty$. In such a case, the angle between the lines will be calculated by drawing the figure.
- The lines are parallel, if the cross product in between the unit vector u and v is zero:

$$u \times v = 0$$

$$\left(\frac{1}{\sqrt{1+m_1^2}}, \frac{m_1}{\sqrt{1+m_1^2}}\right) \times \left(\frac{1}{\sqrt{1+m_2^2}}, \frac{m_2}{\sqrt{1+m_2^2}}\right) = 0$$

$$\frac{m_1 - m_2}{\sqrt{(1+m_1^2)(1+m_2^2)}} k = 0$$

$$m_1 - m_2 = 0 \Rightarrow m_1 = m_2$$

where k is normal to the plane of the lines.

- The lines are perpendicular, if the dot product inbetween the unit vectors u and v is zero:

$$u.v=0$$

$$\left(\frac{1}{\sqrt{1+m_1^2}},\frac{m_1}{\sqrt{1+m_1^2}}\right)\cdot\left(\frac{1}{\sqrt{1+m_2^2}},\frac{m_2}{\sqrt{1+m_2^2}}\right)=0$$

$$\frac{1+m_1m_2}{\sqrt{(1+m_1^2)(1+m_2^2)}}=0$$

$$1+m_1m_2=0$$

$$m_1m_2=-1$$

Example 6.6.1:[Angle between the Lines]: Find the angle from the line $7x+3y-9=0$ to the line $5x-2y+2=0$.

Solution: The slope of a line $7x+3y-9=0$ is

$$3y=-7x+9 \Rightarrow y=-\frac{7}{3}x+3,\ m_1=-\frac{7}{3}$$

The slope of a line $5x-2y+2=0$ is

$$-2y=-5x-2 \Rightarrow y=\frac{5}{2}x+1,\ m_2=\frac{5}{2}$$

If θ is the angle from first line to line second, then

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{-\frac{7}{3}-\frac{5}{2}}{1+\frac{-7}{3}\cdot\frac{5}{2}}=\frac{-29}{6\left(1-\frac{35}{6}\right)}=\frac{-29}{-29}=1\Rightarrow\theta=\tan^{-1}1=45^0$$

The angle $\theta=45^0$ is acute.

ii) *The Equation of family of lines passing through the point of intersection of two lines*

Suppose, the two lines are

$$L_1: a_1x+b_1y+c_1=0 \qquad (42)$$

$$L_2: a_2x+b_2y+c_2=0 \qquad (43)$$

and $P(x_1,y_1)$ is their point of intersection. The given lines L_1 and L_2 are used to obtain a first degree equation of a straight line in x and y:

$$(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0,\ \lambda \text{ is constant} \qquad (44)$$

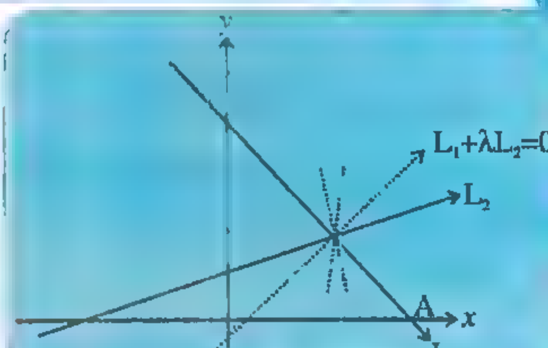


Fig. 6.22

The coordinates of a point P will reduce each line in (44) to zero, since, by hypothesis, P is the point of intersection, i.e., it lies on each line. Therefore P satisfies (44) and represents the **family of lines through the point of intersection of** $L_1 = 0$ **and** $L_2 = 0$.

Example 6.6.2:[Family of Lines]: Develop the family of lines through the point of intersection of the lines $2x-3y+4=0$ and $2x+y-1=0$. Find the line from the family of lines which is

a. parallel to the line whose slope is $m_1 = -2/3$.

b. perpendicular to the line $4x+3y-1=0$.

Solution: Result (44) is used for the lines $2x-3y+4=0$, $2x+y-1=0$ to obtain the family of lines:

$$(2x-3y+4)+\lambda(2x+y-1)=0 \qquad (45)$$

$$(2+2\lambda)x+(-3+\lambda)y+(4-\lambda)=0$$

The slope of the family of lines is:

$$(2+2\lambda)x+(-3+\lambda)y+(4-\lambda)=0$$

$$(-3+\lambda)y=-(2+2\lambda)x-(4-\lambda)$$

$$y=\frac{-(2+2\lambda)}{-3+\lambda}x-\frac{4-\lambda}{-3+\lambda},\ m_2=\frac{-(2+2\lambda)}{-3+\lambda}$$

a. The family of lines (45) is parallel to the line with slope $m_1 = -2/3$ if and only if their slopes are equal:

$$\frac{-(2+2\lambda)}{-3+\lambda}=-\frac{2}{3}$$

$$6+6\lambda=-6+2\lambda$$

$$4\lambda=-12 \Rightarrow \lambda=-3$$

The value of $\lambda=-3$ is used in (45) to obtain the particular line from the family of lines (45):

$$(2x-3y+4)-3(2x+y-1)=0$$

$$2x-3y+4-6x-3y+3=0$$

$$-4x-6y+7=0$$

$$4x+6y-7=0$$

b. The slope of the given line $4x+3y-1=0$ is $m_3=-4/3$. The family of lines (45) is perpendicular to the line $4x+3y-1=0$, if and only if the product of their slopes equals -1:

$$\left[\frac{-(2+2\lambda)}{-3+\lambda}\right]\left(-\frac{4}{3}\right)=-1$$

$$\frac{8+8\lambda}{-9+3\lambda}=-1 \Rightarrow 8+8\lambda=9-3\lambda \Rightarrow 11\lambda=1 \Rightarrow \lambda=\frac{1}{11}$$

The value of $\lambda=1/11$ is used in (45) to obtain the particular line from the family of lines:

$$(2x-3y+4)+\frac{1}{11}(2x+y-1)=0$$
$$22x-33y+44+2x+y-1=0$$
$$24x-32y+43=0$$

iii) Angles of the triangle

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle ABC and the slopes of the sides AB, BC and CA of the triangle ABC are respectively:

$m_1=\dfrac{y_2-y_1}{x_2-x_1}$, *slope of side AB*, $m_2=\dfrac{y_3-y_2}{x_3-x_2}$, *slope of side BC*

$m_3=\dfrac{y_1-y_3}{x_1-x_3}$, *slope of side CA*

If θ_1, θ_2 and θ_3 are the angles in between their sides AB to AC, BC to BA and CB to CA respectively, then the angles can be found through results (41):

The angles from the sides AB to AC, BC to BA, and CA to CB of a triangle ABC are respectively:

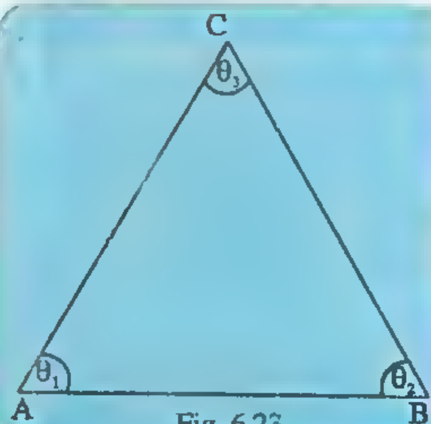


Fig. 6.23

$$\tan\theta_1=\frac{m_1-m_3}{1+m_1m_3},\qquad \tan\theta_2=\frac{m_2-m_1}{1+m_1m_2},\qquad \tan\theta_3=\frac{m_3-m_2}{1+m_2m_3} \qquad (46)$$

Example 6.6.3:[Angles of the Triangle]: Find the angles of the triangle ABC, whose vertices are A (–2,–3), B(4,–1) and C(2,3).

Solution: If ABC is a triangle and the slopes of their sides AB, BC and CA are respectively:

$$m_1=\frac{y_2-y_1}{x_2-x_1}=\frac{-1+3}{4+2}=\frac{2}{6}=\frac{1}{3},\quad \textit{slope of side AB}$$

$$m_2=\frac{y_3-y_2}{x_3-x_2}=\frac{3+1}{2-4}=-\frac{4}{2}=-2,\quad \textit{slope of side BC}$$

$$m_3=\frac{y_1-y_3}{x_1-x_3}=\frac{-3-3}{-2-2}=\frac{-6}{-4}=\frac{3}{2},\quad \textit{slope of side CA}$$

Result (46) is used to obtain the angle θ_1 from the sides AB to AC:

$$\tan\theta_1=\frac{m_1-m_3}{1+m_1m_3}$$

$$=\frac{\frac{1}{3}-\frac{3}{2}}{1+\left(\frac{1}{3}\right)\left(\frac{3}{2}\right)}$$

$$=\frac{2-9}{6\left(1+\frac{1}{2}\right)}=-\frac{7}{6\left(\frac{3}{2}\right)}=-\frac{7}{9}\Rightarrow\ \theta_1=\tan^{-1}\left(\frac{-7}{9}\right)$$

Result (46) is used to obtain the angle θ_2 from the sides BC to BA:

$$\tan\theta_2=\frac{m_2-m_1}{1+m_1m_2}$$

$$=\frac{-2-\frac{1}{3}}{1+(-2)\left(\frac{1}{3}\right)}=\frac{-7}{3\left(\frac{1}{3}\right)}=-7\Rightarrow\ \theta_2=\tan^{-1}(-7)$$

Result (46) is used o obtain the angle θ_3 from the sides CA to CB:

$$\tan\theta_3=\frac{m_3-m_2}{1+m_2m_3}$$

$$=\frac{\frac{3}{2}+2}{1+(-2)\left(\frac{3}{2}\right)}=\frac{7}{2(1-3)}=\frac{7}{-4}=\frac{7}{-4}\Rightarrow\ \theta_3=\tan^{-1}\left(\frac{-7}{4}\right)$$

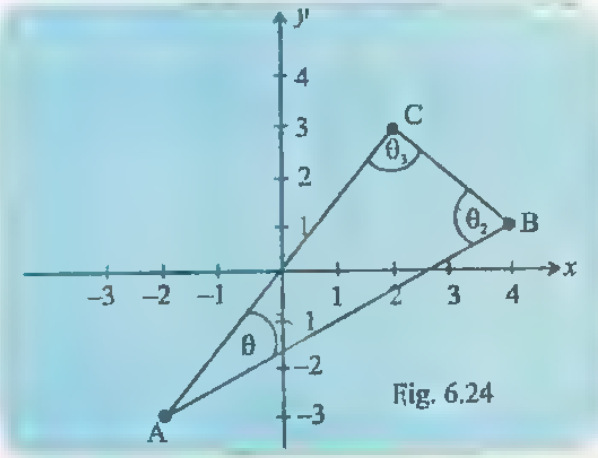


Fig. 6.24

Exercise 6.3

1. In each case, show that the point $P(x_1, y_1)$ lies above or below the line ax+by+c=0. Also show that the point $P(x_1, y_1)$ and the origin lie on the same side or on the opposite side of the line ax+by+c=0:

   a. P(4,–5), 4x–3y–17=0   b. P(–3,8), 5x+7y+9=0

   c. P(20,–15), 6x+9y+13=0

2. In each case, show that the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are on the same side or on the opposite side of the line ax+by+c=0:

   a. P(–4,2), Q(11,–3); 5x+14y–11=0

   b. P(–3,5), Q(1,–2); 2x–3y–10=0

   c. P(–3,2),Q(4,5); 3x+7y–15=0

3. In each case, find the perpendicular distance from the line ax+by+c=0 to a point $P(x_1, y_1)$:

   a. P(3,–4), 4x–3y+6=0   b. P(5,8), 3x–2y+7=0

   c. P(3,–1), 5x+12y–16=0   d. P(–54,71), 4x+3y–17=0

   e. P(–60,25), 3x+7y+5=0

4. In each case, find the angle θ from the line L_1 to line L_2, if the slopes of the lines L_1 and L_2 are the following:

   a. $L_1 : m_1 = 1/2$, $L_2 : m_2 = 3$ b. $L_1 : m_1 = 2$, $L_2 : m_2 = 3$

   c. $L_1 : m_1 = -3$, $L_2 : m_2 = 8$ d. $L_1 : m_1 = 0$, $L_2 : m_2 = 2$

5. In each case, find the angle θ from the line L_1 to the line L_2 :

a. L_1 : joins (1,2) and (7,-1), L_2 : joins (3,2) and (5,6).

b. L_1 : joins (2,7) and (7,10), L_2 : joins (1,1) and (5,3).

Try to obtain acute angles.

6. In each case, find the angle θ from the line L_1 to the line L_2 :

a. $L_1 : x-2y+3=0,\quad L_2 : 3x-y+7=0$

b. $L_1 : 2x+4y-10=0,\ L_2 : 5x-3y+1=0$

c. $L_1 : 2x+y-8=0,\quad L_2 : 3x+2y-2=0$

d. $L_1 : 3x-5y+22=0,\ L_2 : 6x+y-22=0$

7. In each case, find the angles of the triangle ABC whose vertices are the following:

a. A(1,2), B(4,2) and C(–2,3)

b. A(3,–4), B(1,5) and C(2,–4).

c. A(–4,0), B(2,0) and C(2,5)

d. A(–1,–1), B(–1,4) and C(2,0).

8. Find the equation of the straight line from the family of straight lines through the point of intersection of the lines

a. $2x-3y+4=0$, $3x+4y-5=0$ and is perpendicular to the line $6x-7y-18=0$.

b. $3x-4y+1=0$, $5x+y-1=0$ and cuts off equal intercepts from the axes.

c. $x-2y=a$, $x+3y=2a$ and is parallel to the line $3x+4y=0$.

d. $2x-y=0$, $3x+2y=0$ and is perpendicular to the line $3x+y-6=0$.

9. Find the equation of the straight line from the family of straight lines through the pointof intersection of the lines

a. $2x+y+1=0$, $2x+3y+5=0$ that touches the point of intersection of $x-y=0$ and $x+y=0$.

b. $2x-3y-4=0$, $3x+y+5=0$ that touches the point of intersection of $x+y-3=0$ and $x-5y+1=0$.

6.7 Concurrency of Straight Lines

Before to touch the concurrency of straight lines, we need to develop the concept of intersection of lines. Logically, the solution of the system of lines exists only, if the **lines intersect**. For illustration, the two lines

$x+y=1$ and $x-y=0$

is forming the system of two linear equations

$$x+y=1$$
$$x-y=0$$

that in matrix form is represented by Ax=b:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The augmented matrix of the system Ax=b is

$$A/b=\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

reduced in an echelon form through row operations

$$A/b=\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \end{pmatrix} \text{ by } R_2+R_1(-1)$$

that gives the system of equations:

$$\left.\begin{matrix} x+y=1 \\ -2y=-1 \end{matrix}\right\}, y=1/2, x=1/2$$

The second equation is giving y=1/2 which is used in first equation to obtain x=1/2. The solution set (x,y) =(1/2,1/2) of the system of two linear equations is **unique** (one solution set). This unique solution is the unique point of intersection at which the given two lines intersect.

It is important to note that the system of two lines

- $x+y=1$ and $x-y=0$ is giving a **uniqe** solution set, since the lines are intersecting at just a single point.
- $x+y=1$ and $x+y=0$ is not giving a solution set, since the lines are not intersecting, because the line are **parallel**.
- $x+y=1$ and $2x+2y=2$ is giving an **infinite** set of solutions, since the lines are intersecting more than one points, because the lines make a sense of coincident lines.

i) *Condition of concurrency of three straight lines*

The condition of concurrency of three straight lines is **the point of intersection at which the three straight lines intersect**. For illustration, if the given three lines are

$$\begin{cases} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \\ a_3x+b_3y+c_3=0 \end{cases} \quad (47)$$

then, the three lines develop a homogeneous system of three linear equations

$$Ax=0$$

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (48)$$

In homogeneous system of three linear equations lines (48), the homogeneous coordinates are used:

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad (49)$$

Concurrency means that the three lines must intersect at a point G(*x*,*y*), say, that can be found by solving the system of linear equations (47). The system (48) has a **nontrivial solution** if the determinant of the coefficient matrix A of the system (48) is zero:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0 \quad (50)$$

This is the **condition of concurrency** of three lines. For required point of concurrency (intersection), the steps involved are the following:

1. Choose any two lines from the given three system of linear equations (47).
2. Develop the system of these two linear equations.
3. Develop the augmented matrix A/b of the system of two linear equations and reduce it in an echelon form to obtain the point of intersection.
4. Substitute the developed point of intersection in the remaining third line. If the point of intersection satisfies the remaining third line, then that point of intersection should be taken as the point of concurreny of the given three lines.

Example 6.7.1:[Concurrency of Lines]: Show that the three lines $x+4y+3=0$, $5x-4y-5=0$ and $2x+2y+1=0$ are concurrent. If the lines are concurrent, then find out the point of concurrency.

Solution: The condition of concurrency (50) in light of the given three lines is going to be zero:

$$\begin{vmatrix} 1 & 4 & 3 \\ 5 & -4 & -5 \\ 2 & 2 & 1 \end{vmatrix} = 1(-4+10)-4(5+10)+3(10+8)=6-60+54=0$$

The given three lines are concurrent. For the point of concurrency $G(x,y)$, choose the first two lines

$$x+4y=-3$$
$$5x-4y=5$$

that develops the system of two linear equations, whose augmented matrix A/b is reduced in an echelon form

$$A/b=\begin{pmatrix} 1 & 4 & -3 \\ 5 & -4 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -3 \\ 0 & -24 & 20 \end{pmatrix} \text{ by } R_2(-5)=R_2+R_1(-5)$$

to obtain the reduced system of linear equations:

$$x+4y=-3$$
$$-24y=20$$

The second equation is giving $y=-5/6$ which is used in first equation to obtain $x=1/3$. The third line with substitution of the point of intersection $(x,y)=(1/3,-5/6)$ is going to be zero:

$2x+2y+1=0 \Rightarrow 2(1/3)+2(-5/6)+1=(2/3)-(5/3)+1=0$

Thus, the given three lines are concurrent at a point $G(1/3,-5/6)$.

ii) *Equations of median, altitude and right bisectors of a triangle*

- ***Concurrency of the right bisectors of a triangle***

To show the concurrency of the right bisectors of a triangle, the procedure developed is as under:

Let ABC be a triangle, whose vertices are $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$. D, E, F are the midpoints of the sides BC, CA, AB of a triangle ABC whose coordinates are respectively:

$$D\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right),\quad E\left(\frac{x_1+x_3}{2},\frac{y_1+y_3}{2}\right),\quad F\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

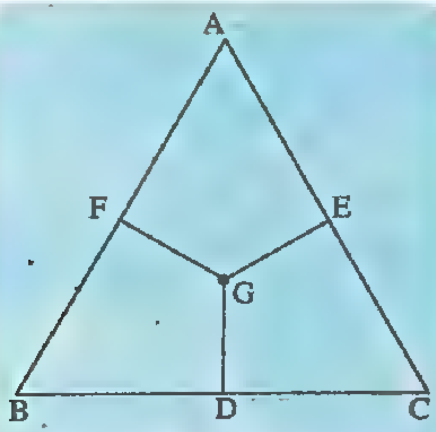


Fig. 6.25

If the slope of the side BC and the slope of the right bisector DG of the side BC are respectively:

$$m_1=\frac{y_3-y_2}{x_3-x_2},\quad m_2=\frac{-1}{m_1}=-\frac{x_3-x_2}{y_3-y_2}$$

then, the equation of the right bisector DG of side BC is obtained by point–slope form of a line:

$$\left(y-\frac{y_2+y_3}{2}\right)=-\frac{x_3-x_2}{y_3-y_2}\left(x-\frac{x_2+x_3}{2}\right)$$

$$(y_3-y_2)\left(y-\frac{y_2+y_3}{2}\right)=(x_2-x_3)\left(x-\frac{x_2+x_3}{2}\right)$$

$$x(x_2-x_3)+y(y_2-y_3)-\frac{1}{2}(y_2^2-y_3^2)-\frac{1}{2}(x_2^2-x_3^2)=0$$

Similarly, the equations of the right bisectors EG (of side CA), FG (of side AB) is respectively:

$$x(x_3-x_1)+y(y_3-y_1)-\frac{1}{2}(y_3^2-y_1^2)-\frac{1}{2}(x_3^2-x_1^2)=0$$

$$x(x_1-x_2)+y(y_1-y_2)-\frac{1}{2}(y_1^2-y_2^2)-\frac{1}{2}(x_1^2-x_2^2)=0$$

The right bisectors DG, EG and FG is concurrent, if the determinant of the coefficient matrix A of the related system of equations of the right bisectors DG, EG and FG equals zero:

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -\frac{1}{2}(y_2^2-y_3^2)-\frac{1}{2}(x_2^2-x_3^2) \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}(y_3^2-y_1^2)-\frac{1}{2}(x_3^2-x_1^2) \\ x_1-x_2 & y_1-y_2 & -\frac{1}{2}(y_1^2-y_2^2)-\frac{1}{2}(x_1^2-x_2^2) \end{vmatrix} \qquad (51)$$

The operation of addition of rows $R_1+R_2+R_3$ to row R_1 is used to obtain:

$$\begin{vmatrix} 0 & 0 & 0 \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}(y_3^2-y_1^2)-\frac{1}{2}(x_3^2-x_1^2) \\ x_1-x_2 & y_1-y_2 & -\frac{1}{2}(y_1^2-y_2^2)-\frac{1}{2}(x_1^2-x_2^2) \end{vmatrix}=0 \qquad (52)$$

The value of the determinant is zero. Hence, the right bisectors DG, EG and FG of a triangle ABC are concurrent.

Example 6.7.2:[Concurreny of the Right Bisectors]: Let ABC be a triangle with vertices A(0,0), B(8,6) and C(12,0). Show that the right bisectors DG, EG and FG of the triangle ABC are concurrent.

Solution: The vertices $A(x_1,y_1)=A(0,0)$, $B(x_2,y_2)=B(8,6)$ and $C(x_3,y_3)=C(12,0)$ of the triangle ABC are used in the determinant (51) to obtain:

$$\begin{vmatrix} -4 & 6 & 22 \\ 12 & 0 & -72 \\ -8 & -6 & 50 \end{vmatrix} = -4(-432)-6(600-576)+22(-72)$$

$$=1728-1728=0$$

The determinant (51) equals zero. Hence, the right bisectors DG, EG and FG of the triangle ABC are concurrent.

The determinant (51) equals zero. Hence, the right bisectors DG, EG and FG of the triangle ABC are concurrent.

- *Concurrency of the altitudes of a triangle*

Let ABC be a triangle, whose vertices are $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$. The altitudes of the triangle ABC are AD, BE and CF.

If the slope of the side BC and the slope of the altitude AD are respectively

$$m_1 = \frac{y_3 - y_2}{x_3 - x_2}, \quad m_2 = -\frac{1}{m_1} = -\frac{x_3 - x_2}{y_3 - y_2},$$

then, the equation of the altitude AD is obtained by point–slope form of a line:

$$(y - y_1) = -\frac{x_3 - x_2}{y_3 - y_2}(x - x_1)$$

$$(y - y_1)(y_3 - y_2) = (x_2 - x_3)(x - x_1)$$

$$x(x_2 - x_3) + y(y_2 - y_3) - x_1(x_2 - x_3) - y_1(y_2 - y_3) = 0$$

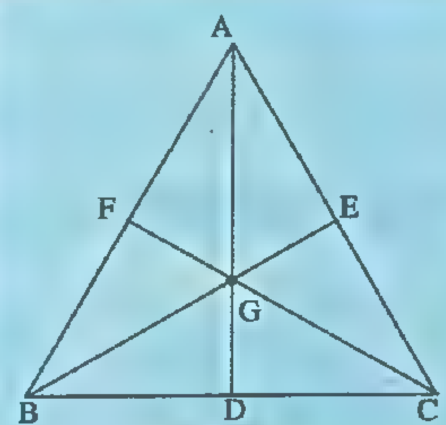


Fig. 6.26

Similarly, the equations of the altitudes BE and CF are respectively:

$$x(x_3-x_1)+y(y_3-y_1)-x_2(x_3-x_1)-y_2(y_3-y_1)=0$$

$$x(x_1-x_2)+y(y_1-y_2)-x_3(x_1-x_2)-y_3(y_1-y_2)=0$$

The altitudes AD, BE and CF are concurrent, if the determinant of the coefficient matrix A of the related system of equations of the altitudes AD, BE and CF equals zero:

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -x_1(x_2-x_3)-y_1(y_2-y_3) \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-y_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix} \qquad (53)$$

The operation of addition of rows $R_1+R_2+R_3$ to row R_1 is used to obtain:

$$\begin{vmatrix} 0 & 0 & 0 \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-y_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix}=0$$

The value of the determinant is zero. Thus, the altitudes AD, BE and CF of a triangle are concurrent at a point G.

The conclusion drawn from the above results is that the three medians AD, BE and CF of a triangle ABC will also make concurrency at a point say, G(x,y).

Example 6.7.3:[Concurrency of the Altitudes]: Let ABC be a triangle with vertices A(0,0), B(8,6) and C(12,0). Show that the altitudes AD, BE and CF of the triangle ABC are concurrent.

Solution: The vertices $A(x_1,y_1)=A(0,0)$, $B(x_2,y_2)=B(8,6)$ and $C(x_3,y_3)=C(12,0)$ of the triangle ABC are used in the determinant (53) to obtain:

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -x_1(x_2-x_3)-y_1(y_2-y_3) \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-y_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix}=\begin{vmatrix} -4 & 6 & 0 \\ 12 & 0 & -96 \\ -8 & -6 & 96 \end{vmatrix}$$

$$=-4(-576)-6(1152-768)$$

$$=2304-6(384)=2304-2304=0$$

The determinant (53) equals zero. Hence, the altitudes AD, BE and CF of the triangle ABC are concurrent.

6.8 Area of a Triangular Region

Let ABC be a triangle whose vertices are $P_1(x_1, y_1), P_2(x_2, y_2)$ and $P_3(x_3, y_3)$. Project P_1A, P_2B and P_3C upon the x-axis that develops the trapezia P_1ACP_3, P_3CBP_2 and P_1ABP_2.

The area A of the triangular region $P_1P_2P_3$ is the sum of the areas of the trapezia P_1ACP_3, P_3CBP_2 and P_1ABP_2 minus the area of the trapezium P_1ABP_2:

$$A=\frac{1}{2}[(y_1+y_3)(x_3-x_1)]+\frac{1}{2}[(y_3+y_2)(x_2-x_3)]-\frac{1}{2}[(y_1+y_2)(x_2-x_1)]$$

$$=\frac{1}{2}[x_3y_1-x_1y_1+x_3y_3-x_1y_3+x_2y_3-x_3y_3+x_2y_2-x_3y_2$$

$$-x_2y_1+x_1y_1-x_2y_2+x_1y_2] \qquad (54)$$

$$=\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

$$=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \text{ first column expansion}$$

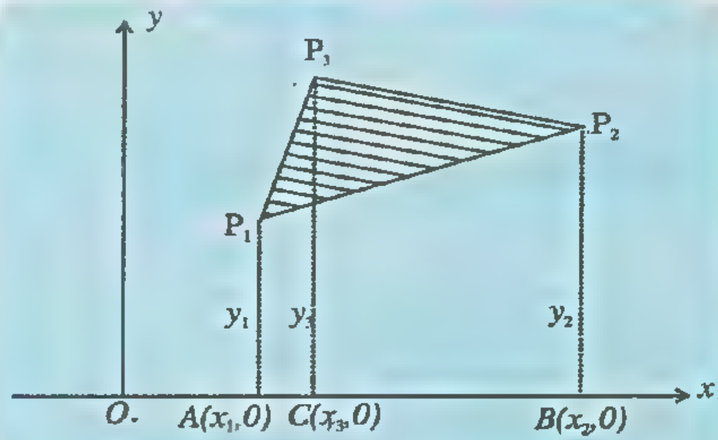


Fig. 6.27

It is important to note that the area A of the triangular region $P_1P_2P_3$ equals zero, when the vertices of the triangular region are collinear points.

Example 6.8.1:[Area of Triangular Region]: Find the area of the triangular region $P_1P_2P_3$ whose vertices are $P_1(4,-5)$, $P_2(5,-6)$ and $P_3(3,1)$.

Solution: Result (54) is used for the vertices of the triangular region $P_1(4,-5)$, $P_2(5,-6)$, $P_3(3,1)$ to obtain the area of the triangular region $P_1P_2P_3$:

$$A=\frac{1}{2}\begin{vmatrix}4 & -5 & 1\\ 5 & -6 & 1\\ 3 & 1 & 1\end{vmatrix},\ \textit{first row expansion}$$

$$=\frac{1}{2}[4(-6-1)+5(5-3)+(5+18)]$$

$$=\frac{1}{2}[-28+10+23]$$

$$=\frac{5}{2}\ \textit{square units}$$

6.9 Homogeneous Equation

In general, any line equation in two variables that passes through the origin is called a **homogeneous equation**.

i) *Homogeneous linear and quadratic equations in two variables*

Definition 6.9.1:[Homogeneous & Nonhomogeneous Linear equations]: An equation of the form in two variables x and y

$$ax+by+c=0,\ c\neq 0,\ \text{a,b and c are constants} \qquad (55)$$

is called a nonhomogenous equation of a line. For c=0, the nonhomogeneous equation (55) gives the homogeneous equation of the form

$$ax+by=0 \qquad (56)$$

that passes through the origin definitely.This also defines a homogeneous equation of degree 1, since the indices of x and y in every-term of (56) is the same, the degree being 1. For example, the equation of line x+y=0 is homogeneous line, since it defines a homogeneous equation of degree 1.

Definition 6.9.2: [Homogeneous Quadratic Equation]: An equation of the form

$$ax^2 + 2hxy + by^2 = 0, \quad a \neq 0, \quad \text{a,b,c are constants} \qquad (57)$$

is called a homogeneous quadratic equation of second degree in variables x and y, since the sum of the indices of x and y in every term are the same number "2." For example,

$$3x^2 - 4xy + 5y^2 = 0 \qquad \text{and} \qquad lx^2 + mxy + ny^2 = 0$$

are homogeneous quadratic equations of the second degree in x and y. On the other hand, the equation of the form $3xy^2 - 4xy + 5y^2 = 0$ is not a homogeneous equation, since the sum of the indices of x and y are not the same in each and every term.

ii) Second degree homogeneous equation represents a pair of straight lines through the origin

- *Standard form of second degree homogeneous equation*

If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are the two straight lines, then the simple product of the given two nonhomogeneous lines defines a joint equation of a line:

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0 \qquad (58)$$

The **joint equation of the homogeneous straight lines** is obtained from (58) by putting $c_1 = c_2 = 0$:

$$(a_1x + by_1)(ax_2 + by_2) = 0 \qquad (59)$$

The product of homogeneous lines (59) is giving the **standard form of the second degree homogeneous equation**:

$$(a_1x + b_1y)(a_2x + b_2y) = 0$$

$$a_1a_2x^2 + a_1b_2xy + b_1a_2xy + b_1b_2y^2 = 0 \qquad (60)$$

$$a_1a_2x^2 + (a_1b_2 + a_2b_1)xy + b_1b_2y^2 = 0$$

If $a_1a_2 = a$, $(a_1b_2 + a_2b_1) = 2h$, $b_1b_2 = b$, then (60) gives:

$$ax^2 + 2hxy + by^2 = 0 \qquad (61)$$

Any point P(x,y) that satisfies first line $a_1x + b_1y = 0$ or second line $a_2x + b_2y = 0$ will also satisfies the joint homogeneous equation of (59).

- ***Representation as pair of straight lines***

The product of equation (61) to constant quantity a/a is giving the joint equation of the two first degree homogeneous equations in x and y:

$$\frac{a}{a}\left[ax^2+2hxy+by^2\right]=0$$

$$a^2x^2+2ahxy+aby^2=0, \textit{ Add and subtract } h^2y^2$$

$$(ax+hy)^2-h^2y^2+aby^2=0$$

$$(ax+hy)^2-y^2(h^2-ab)=0$$

$$(ax+hy)^2-\left(y\sqrt{(h^2-ab)}\right)^2=0$$

$$\left(ax+hy+y\sqrt{h^2-ab}\right)\left(ax+hy-y\sqrt{h^2-ab}\right)=0$$

$$ax+hy+y\sqrt{h^2-ab}=ax+\left(h+\sqrt{h^2-ab}\right)y=0 \qquad (62)$$

$$ax+hy-y\sqrt{h^2-ab}=ax+\left(h-\sqrt{h^2-ab}\right)y=0 \qquad (63)$$

The lines (62) and (63) are therefore first degree equations in x and y.

It is important to note that the lines are

- real and distant, if $h^2-ab>0$.
- real and coincident, if $h^2-ab=0$.
- imaginary, if $h^2-ab<0$.

Example 6.9.1:[Joint Straight Lines]: Find two first degree straight lines in x and y when the second degree homogeneous equation is $5x^2+3xy-8y^2=0$.

Solution: The standard form of second degree homogeneous equations (61) is compared to the given second degree homogeneous equation $5x^2+3xy-8y^2=0$ to obtain:

$$a=5,\ 2h=3\Rightarrow h=3/2,\ b=-8$$

These values are used in the standard form of two first degree homogeneous lines (62) and (63) to obtain the required two homogeneous lines:

The two lines are perpendicular, since a+b=3–3=0 is zero for a=3 and b= –3.

Example 6.9.4:[Joint Equation]: Find a joint equation of the straight line that passes through the origin and perpendicular to the lines represented by $2x^2-5xy+3y^2=0.$

Solution: The standard form of second degree homogeneous equations (61) is compared to the given second degree homogeneous equation $2x^2-x5y+3y^2=0$ to obtain:

$$a=2,\ 2h=-5 \Rightarrow h=-5/2,\ b=3$$

These values are used in the standard forms (62) and (63) of the two first degree homogeneous lines to obtain the required two homogeneous lines:

$$ax+\left(h+\sqrt{h^2-ab}\right)y=0 \qquad ax+\left(h-\sqrt{h^2-ab}\right)y=0$$

$$2x+\left[-\frac{5}{2}+\sqrt{\frac{25}{4}-(2)(3)}\right]y=0 \qquad 2x+\left[-\frac{5}{2}-\sqrt{\frac{25}{4}-(2)(3)}\right]y=0$$

$$2x+\left[-\frac{5}{2}+\sqrt{\frac{25}{4}-6}\right]y=0 \qquad 2x+\left[-\frac{5}{2}-\sqrt{\frac{25}{4}-6}\right]y=0$$

$$2x+\left[-\frac{5}{2}+\frac{1}{2}\right]y=0 \qquad 2x+\left[-\frac{5}{2}-\frac{1}{2}\right]y=0$$

$$2x-2y=0 \qquad 2x-3y=0$$

$$x-y=0$$

Slope of the first line is

$$x-y=0$$
$$y=x,\ m_1=1$$

Slope of the second line is

$$2x-3y=0$$
$$-3y=-2x$$
$$y=\frac{2}{3}x,\ m_2=2/3$$

The line that passes through the origin O(0,0) and perpendicular to line first is:

$$y-0=\frac{-1}{m_1}(x-0) \Rightarrow y=-x \Rightarrow x+y=0 \qquad (67)$$

The line that passes through the origin O(0,0) and perpendicular to line second is:

$$y-0=\frac{-1}{m_2}(x-0) \Rightarrow y=-\frac{3}{2}x \Rightarrow 3x+2y=0 \qquad (68)$$

The joint second degree equation of the two lines (67) and (68) is:

$$(x+y)(3x+2y)=0$$
$$3x^2+5xy+2y^2=0$$

Exercise 6.4

1. In each case, find the point of intersection P(x, y) of the pair of lines:

   a. $2x+4y-10=0,\ 5x-3y+1=0$

   b. $2x+y-8=0,\ 3x+2y-2=0$

   c. $3x-5y+22=0,\ 6x+y-22=0$

   d. $8x-7y+1=0,\ 10x-11y+35=0$

2. Show that the following lines are concurrent. If the lines are concurrent, then find out the point at which the given lines can make concurrency:

   a. $x-y-2=0,\quad 2x-y-5=0,\quad 11x-5y-28=0$

   b. $x+2y-3=0,\quad 2x-y+4=0,\quad x+4y-7=0$

   c. $3x+2y-1=0,\quad 2x-3y+4=0,\quad x+y-2=0$

   d. $x+2y+5=0,\quad 3x+5y+1=0,\quad 4x+7y+6=0$

3. If ABC is a triangle with vertices A(0,0), B(8,6) and C(12,0), then show that

   a. the right bisectors of the triangle ABC are concurrent.

   b. the altitudes of the triangle ABC are concurrent.

   c. the medians of the triangle ABC are concurrent.

4. Find the area of the triangular region whose vertices are the following:

   a. $P_1(0,0), P_2(2,4), P_3(-2,2)$. b. $P_1(-1,-2), P_2(2,5), P_3(5,2)$.

   c. $P_1(4,-5), P_2(5,-6), P_3(3,1)$.

5. Find the area of the region bounded by the triangle ABC whose vertices are A(a, b+c), B(a, b–c) and C(–a, c).

6. Show that the area bounded by the triangle ABC whose vertices are the following:

a. A(–3, 6), B(3, 2) C(6, 0) b. A(–2, 4), B(3, –6) C(1, –2)

c. Are the vertices in parts a and b of the triangle ABC collinear?

7. Find two first degree straight lines in x and y, when the second degree homogeneous equations are the following:

a. $3x^2-2xy-5y^2=0$ b. $4x^2-9xy+5y^2=0$

c. $x^2+xy+y^2=0$ d. $x^2-7xy+6y^2=0$

8. Find the angles in between the lines represented by the following second degree homogeneous equations:

a. $3x^2-2xy-5y^2=0$ b. $4x^2-9xy+5y^2=0$

c. $x^2+xy+y^2=0$ d. $x^2-7xy+6y^2=0$

9. Show the two first degree straight lines in x and y are coincident, perpendicular or neither, when they are represented by the following second degree homogeneous equations:

a. $x^2+5xy-y^2=0$ b. $2x^2-xy-y^2=0$

c. $x^2+3xy+9y^2=0$ d. $x^2+4xy+8y^2=0$

10. Find a joint equation of the straight line that passes through the origin and

a. perpendicular to the lines represented by $3x^2-7xy+2y^2=0$.

b. perpendicular to the lines represented by $x^2-2\tan\theta xy-y^2=0$.

c. perpendicular to the lines represented by $ax^2+2hxy+by^2=0$.

Glossary

- The distance from point $P(x_1, y_1)$ to point $Q(x_2, y_2)$ in the coordinate plane is:

  $d = |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

- If L_1 and L_2 are the two lines having slopes m_1 and m_2, then, these two lines are

  a. parallel if and only if they have the same slopes: $m_1 = m_2$.

  b. perpendicular if and only if the product of their slopes equals -1: $m_1 m_2 = -1$

- The equation of a straight line parallel to the x–axis and at a distance a from it, is y=a. The equation of the x–axis is y=0 and the vector equation of x–axis is r.j=0.
- The equation of a straight line parallel to the y–axis and at a distance b from it, is x=b. The equation of the y–axis is x=0 and the vector equation of y–axis is r.i=0.
- The standard forms of the line are the following:

a. $y=mx+c$,	Slope–Intercept form
b. $y - y_1 = m(x - x_1)$,	Point–Slope form
c. $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$	Two–Point form
d. $\frac{x}{a} + \frac{y}{b} = 1$	Double–Intercept form
e. $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$	Symmetric Form
f. $x\cos\theta + y\sin\theta = p$	Normal Form

- The standard form of a line is $ax+by+c=0$
- For c=0, the line is homogeneous that passes through the origin.

- For $c \neq 0$, the line is nonhomogeneous that does not pass through the origin.
- The perpendicular distance from a line ax+by+c=0 to a point $P(x_1, y_1)$ is:

$$d = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

- The angle between the two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is:

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

- The general equation of a straight line that passes through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is:

$$(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0, \quad \lambda \text{ is constant}$$

- The condition of concurrency of the three lines is:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

- An equation of the form

$$ax^2 + 2hxy + by^2 = 0, \quad a \neq 0, b, c \text{ are constants}$$

is called a homogeneous equation of second degree in x and y when the sum of the indices of x and y in every term is the same, the sum being 2.

- The angle between two homogeneous straight lines $y = m_1x$ and $y = m_2x$ is:

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

  - The given two straight lines are perpendicular, if the angle between them is 90° that makes a+b=0.
  - The given two straight lines are coinciding if the angle between them is zero that makes $h^2 = ab$.

7 CONICS-I

This unit tells us, how to:

- define conics and the members of its family as circle, parabola, ellipse and hyperbola.
- define the circle and its equation in standard form.
- recognize the general equation of a circle and its centre and radius.
- find the equation of a circle determined by a given condition
- find the condition when a line intersects the circle and touches the circle.
- find the equation of a tangent to a circle in slope form
- find the equations of a tangent and a normal to a circle at a given point.
- find the length of tangent to a circle from a given external point.
- find two tangents drawn to a circle from an external point that are equal in length.
- prove the properties of a circle such as perpendicular from the center of a circle on a chord that bisects the chord, perpendicular bisector of any chord of a circle passes though the center of the circle, line joining the center of a circle are equidistant from its center and its converse, measure of the central angle of a minor arc is double the measure of the angle subtended by the corresponding major arc, an angle in semicircle is a right angle, the perpendicular at the outer end of a radial segment is tangent to the circle and the tangent to a circle at any point of the circle is perpendicular to the radial segment at that point.

7.1 Introduction

The conic sections have been studied extensively since ancient times and have many important applications. For example, in the early 17^{th} century, Johannes Kepler observed that the planets travel in elliptical paths and Galileo discovered that in a vacuum, a projectile follows a parabolic path. At the end of the 17^{th} century, Newton used the fact that planets follow elliptical paths as the basis for the inverse-square law of gravitational attraction. In more modern times,

conic sections have been used in architecture, in the design of lenses and mirrors, and to study the paths of atomic particles.

i) *Conics and members of its family*

Mathematically, the general second degree equation

$$Ax^2 + Bxy + Cy^2 + DX + EY + F = 0, \qquad (1)$$

is the equation of a circle, a parabola, an ellipse, or a hyperbola depending on the values of A, B and C. If A = B = C = 0, then the equation is not quadratic, but linear equation represents a straight line. Historically, second degree equations in two variables were first considered in a geometric context and were called conic sections, because the curves, they represent can be described as the intersection of a double-napped right circular cone and plane as shown in Fig. 7.1:

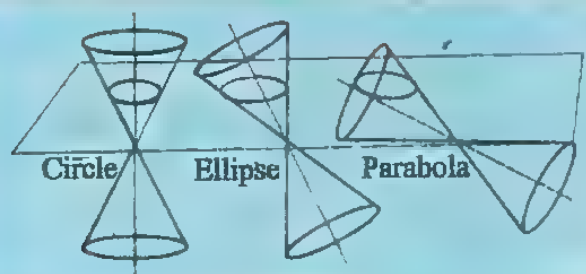


Fig 7.1

In this unit, we look at one of these conic sections that are the circles.

7.2 Circle

A circle is a shape that has a continuous and constant curve. Though it is always curving; it has an algebraic expression that describes its nature.

7.2.1 *Equation of a circle*

i) *Definition of circle equation and its derivation in standard form*

Definition 7.2.1:[Circle]: The set of all points in the plane in such a way that its distances from a fixed point in that plane **(called the center)** is equal to a fixed distance **(called the radius)** of the circle.

Derivation of Circle Equation: This definition helps us in developing a standard form of the equation of a circle. The procedure developed is as under:

Let C (h, k) be Centre of the circle and r is the radius of the circle and P(x, y) is any one of the collection of points on the circumference of the circle that gives the distance from the fixed point C(h, k) which is called the radius of a circle. The position vectors of P and C relative to origin are respectively.

$$OP=(x,y),\ OC=(h,k) \qquad (2)$$

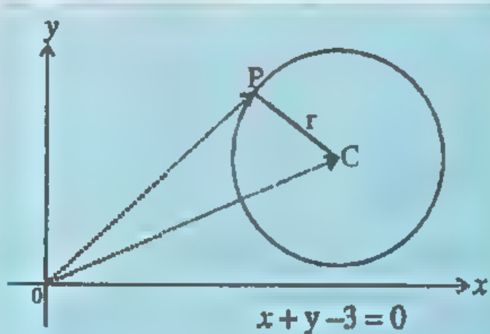


Fig. 7.2

From the Fig. 7.2, the distance from the center C to point P is the fixed distance equals the radius of the circle:

$$OC+CP=OP$$

$$CP=OP-OC$$

$$CP=(x,y)-(h,k)$$

$$=(x-h,y-k)$$

$$|CP|=\sqrt{(x-h)^2+(y-k)^2}, \qquad \text{distance formula}$$

$$|CP|^2=\left(\sqrt{(x-h)^2+(y-k)^2}\right)^2, \qquad \text{squaring both sides}$$

$$r^2=(x-h)^2+(y-k)^2, \qquad |CP|^2=(CP)^2=r^2$$

Definition 7.2.2:[Standard Form of the Equation a Circle]: The standard form of the equation of a circle with radius r and centre (h, k) is:

$$(x-h)^2+(h-k)^2=r^2 \qquad (3)$$

If the center of the circle is at the origin (h,k)=(0,0), then the circle equation (3) becomes:

$$x^2+y^2=r^2 \qquad (4)$$

Example 7.2.1: [Circle with Radius and Center]: Determine the equation of a circle with center at (–2,1) and radius r = 3.

Solution: Result (3) for center C(h, k) = (–2,1) and radius r = 3 is used to obtain the circle equation:

$$(x-h)^2+(y-k)^2=r^2 \Rightarrow (x+2)^2+(y-1)^2=9,\ h=-2,\ k=1,\ r=3$$

7.2.2 General form of an equation of a circle

ii) Recognition of general equation of a circle and its center and radius

The rearrangement of the general equation of the second degree in x and y that may represent a circle through the following procedure:

The general equation of the second degree in variables x and y is:

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad (5)$$

Divide out both sides of equation (5) by a to obtain:

$$x^2+\frac{2h}{a}xy+\frac{b}{a}y^2+\frac{2g}{a}x+\frac{2f}{a}y+\frac{c}{a}=0 \qquad (6)$$

$$x^2+\frac{2h}{a}xy+\frac{b}{a}y^2+2g_1x+2f_1y+c_1=0,\quad g_1=g/a, f_1=f/a, c_1=c/a$$

The rearranged equation (6) of the general equation of the second degree (5) in x and y gives the general equation of a circle if and only if $b/a=1$ and $2h/a=0$:

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$
$$x^2+y^2+2gx+2fy+c=0,\ g_1=g,\ f_1=f, c_1=c \qquad (7)$$

For its center and radius, add and subtract g^2 and f^2 to equation (7) to obtain

$$x^2+y^2+2gx+2fy+c=0$$

$$x^2+2gx+g^2-g^2+2fy+f^2-f^2+c=0,$$

Add and subtract g^2 and f^2

$$(x+g)^2+(y+f)^2=g^2+f^2-c$$

$$[x-(-g)]^2+[y-(-f)]^2=\left(\sqrt{g^2+f^2-c}\right)^2 \qquad (8)$$

the locus of a point (x, y) which moves in such a way that its distance from a fixed point $(-g,-f)$ is constant and equals $\sqrt{g^2+f^2-c}$. This of course represents a circle.

It is important to note:

- **Center and Radius:** The coordinates of the center are $(-g,-f)$ and the radius is $r=\sqrt{(-g)^2+(-f)^2-c}$.
- **Independent Constant:** The general equation contains three independent constants g, f and c. They can be determined from the three independent conditions.
- **Nature of the Circle:**

  If $g^2+f^2-c>0$, then, the circle is real and different from zero.

  If $g^2+f^2-c=0$, then, the circle shrinks into a point $(-g,-f)$. It is called point circle.

  If $g^2+f^2-c<0$, then, the circle is imaginary or virtual.
- The coefficients of x^2 is equal to the coefficients of y^2, and there is no term containing xy and the square of the radius $r^2 \geq 0$.

Example 7.2.2:[Center and Radius of a Circle]: Find the center and radius of a circle $45x^2+45y^2-60y+36y+19=0$.

Solution: The given circle equation is rearranged to obtain:

$$x^2+y^2-\frac{4}{3}x+\frac{4}{5}y+\frac{19}{45}=0, \quad \text{Divide out by 45} \qquad (9)$$

The circle equation (9) is compared to the general form of a circle (8) to obtain the values of g, f and c:

$$2g=-4/3 \Rightarrow g=-2/3,\ 2f=4/5 \Rightarrow f=2/5,\quad c=19/45$$

The center and radius of the given circle are therefore:

$$(-g,-f)=\left(\frac{2}{3},\frac{-2}{5}\right),\ r=\sqrt{\frac{4}{9}+\frac{4}{25}-\frac{19}{45}}=\sqrt{\frac{41}{225}}=\sqrt{\frac{41}{15}}$$

iii) Equation of a circle determined by a given condition

For the required equation of a circle, we need to find out the values of the three independent constants g, f and c under condition that

1. the circle might be known to pass through three specific non-collinear points.
2. the two points are on the circle and the center to lie on a given straight line.
3. the two points are on the circle and the equation of tangent to the circle is known at one of the two points.
4. the two points are on the circle and is touching a straight line.

Example 7.2.3:[Condition First]: Find the equation of a circle which passes though the three points A (1,0), B(0,–6) and C(3,4).

Solution: The required equation of a circle

$$x^2+y^2+2gx+2fy+c=0 \qquad (10)$$

passes through the three points A(1,0), B(0,–6) and C(3,4) which gives a system of three linear equations in three unknowns g, f and c:

$$\begin{aligned} 1+2g+c&=0 & & 2g+c=-1 \\ 36-12f+c&=0 & \Rightarrow\quad & -12f+c=-36 \qquad (11)\\ 25+6g+8f+c&=0 & & 6g+8f+c=-25 \end{aligned}$$

The system of three linear equation (11) in matrix form

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & -12 & 1 \\ 6 & 8 & 1 \end{pmatrix}\begin{pmatrix} g \\ f \\ c \end{pmatrix}=\begin{pmatrix} -1 \\ -36 \\ -25 \end{pmatrix},\qquad Ax=b$$

whose augmented matrix is:

$$A|b=\begin{pmatrix}2 & 0 & 1 & -1\\ 0 & -12 & 1 & -36\\ 6 & 8 & 1 & -25\end{pmatrix}$$

Reduce this augmented matrix in an echelon form to obtain:

$$\sim\begin{pmatrix}2 & 0 & 1 & -1\\ 0 & -12 & 1 & -36\\ 0 & 8 & -2 & -22\end{pmatrix}, R_{31}(-3)=R_3+R_1(-3)$$

$$\sim\begin{pmatrix}2 & 0 & 1 & -1\\ 0 & -12 & 1 & -36\\ 0 & 0 & -4/3 & -46\end{pmatrix}, R_{32}(2/3)=R_3+\frac{2}{3}R_2$$

$$\begin{aligned}2g+c&=-1\\ -12f+c&=-36 \qquad (12)\\ (-4/3)c&=-46\end{aligned}$$

Third equation of the system (12) is giving c = 69/2 which is used in second and first equations to obtain the values of f = 47/8 and g = −17/4.

The values of g = −17/4, f = 47/8 and c = 69/2 are used in equation (10) to obtain the required circle equation:

$$x^2+y^2+2\left(\frac{-71}{4}\right)x+2\left(\frac{47}{8}\right)y+\frac{69}{2}=0$$

$$x^2+y^2-\frac{71}{2}x+\frac{47}{4}y+\frac{69}{2}=0$$

$$4x^2+4y^2-142x+47y+138=0$$

Example 7.2.4: [Condition Second]: Find the equation of a circle which passes through the points A(3,1) and B(2,2) having its centre on the line $x+y-3=0$.

Solution" Let the required equation of a circle be

$$x^2+y^2+2gx+2fy+c=0 \qquad (13)$$

If the circle equation (13) passes through the two points A(3.1) and B(2,2), then it gives a system of two linear equations

$$\begin{aligned} 10+6g+2f+c&=0 \\ 8+4g+4f+c&=0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 6g+2f+c&=-10 \\ 4g+4f+c&=-8 \end{aligned} \qquad (14)$$

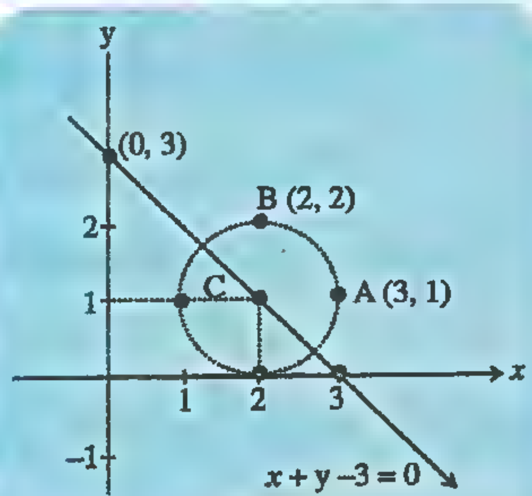


Fig. 7.3

If the center (–g,–f) of the circle lies on the line $x+y-3=0$, then the line $x+y-3=0$ becomes:

$$-g-f-3=0 \;\Rightarrow\; g+f=-3, \quad x=-g, \; y=-f \qquad (15)$$

The combination of equations (14) and (15) is giving the system of three linear equations in three unknown **g, f** and **c**

$$\begin{pmatrix} 6 & 2 & 1 \\ 4 & 4 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} g \\ f \\ c \end{pmatrix}=\begin{pmatrix} -10 \\ -8 \\ -3 \end{pmatrix}, \qquad Ax=b$$

whose augmented matrix is:

$$A|b=\begin{pmatrix} 6 & 2 & 1 & -10 \\ 4 & 4 & 1 & -8 \\ 1 & 1 & 0 & -3 \end{pmatrix}$$

Reduce this augmented matrix in an echelon form to obtain the unknowns **g, f** and **c**:

$$\sim\begin{pmatrix}6 & 2 & 1 & -10\\ 0 & 8/3 & 1/3 & -4/3\\ 0 & 2/3 & -1/6 & -4/3\end{pmatrix}, R_{21}(-2/3), R_{31}(-1/6)$$

$$\sim\begin{pmatrix}6 & 2 & 1 & -10\\ 0 & 8/3 & 1/3 & -4/3\\ 0 & 0 & -1/4 & -1\end{pmatrix}, R_{32}(-1/4)$$

$$6g+2f+c=-10$$

$$(8/3)f+(1/3)c=-4/3 \qquad (16)$$

$$(-1/4)c=-1$$

Third equation of the system (16) is giving c = 4 which is used in second and first equations to obtain the values of f = –1 and g = –2.

The values of g = –2, f = –1 and c = 4 are used in equation (13) to obtain the required circle equation:

$$x^2+y^2-4x-2y+4=0$$

Example 7.2.5:[Condition Third]: Find the equation of a circle which passes through the two points A(0,–1) and B(3,–3) and $3x-2y-2=0$ is the tangent line on the circle at a point A(0,–1).

Solution: Let C (h,k) be the center of the required circle. If A(0,–1) and B(3,–3) are the two points lie on the circle, then the square of the distance from C to A equals the square of the distance from C to B:

$$|CA|^2=|CB|^2, \quad CA=(h-0,\ k+1),\ CB=(h-3,\ k+3)$$

$$(h-0)^2+(k+1)^2=(h-3)^2+(k+3)^2$$

$$h^2+k^2+2k+1=h^2-6h+9+k^2+9+6k$$

$$6h-4k-17=0 \qquad (17)$$

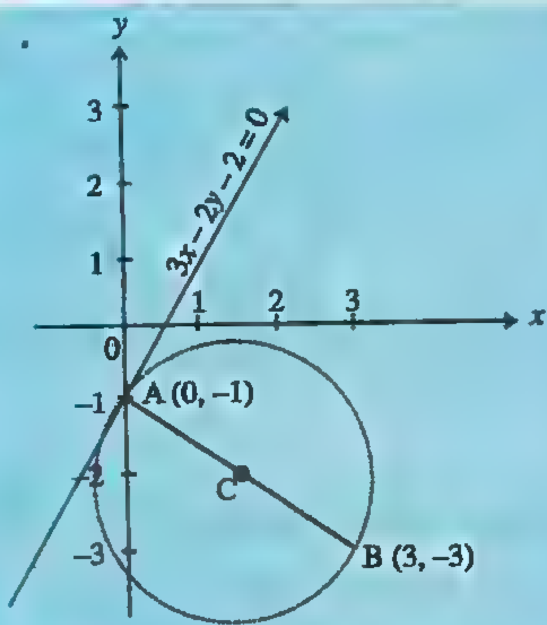


Fig. 7.4

The slope of CA is

$$m_1 = \frac{-1-k}{0-h} = \frac{k+1}{h}$$

and the slope of the tangent line 3x-2y-2=0 is

$$3x-2y-2=0$$

$$-2y=-3x+2 \Rightarrow y=\frac{3}{2}x-1,\ m_2=\frac{3}{2}$$

If CA is perpendicular to the tangent line $3x-2y-2=0$, then the product of their slopes equals −1:

$$m_1 m_2 = -1$$

$$\left(\frac{k+1}{h}\right)\left(\frac{3}{2}\right) = -1$$

$$3k+3=-2h \Rightarrow 2h+3k+3=0 \qquad (18)$$

The equations (17) and (18) are solved to obtain the values of k = −2 and h= 3/2.

The required circle with center (h,k) = (3/2,–2) and radius

$$r=|CA|=|CB|=\sqrt{4^2+(k+1)^2}=\sqrt{(9/4)+1}=\sqrt{13}/2 \text{ is:}$$

$$(x-h)^2+(y-k)^2=r^2$$

$$\left(x-\frac{3}{2}\right)^2+(y+2)^2=\frac{13}{4}\Rightarrow x^2+y^2-3x+4y+3=0$$

Example 7.2.6: [Condition Fourth]: Find the equation of a circle which passes through the two points A(0,0) and B(4,0) and is touching a line $3x+4y+4=0$.

Solution: Let C(h,k) be the center of the required circle. If A(0,0) and B(4,0) are the two points lie on the circle, then the radius of the circle from center C to point A equals the radius of the circle from the center C to point B:

$$|CA|^2=|CB|^2,\ CA=(0-h,0-k),\ CB=(4-h,0-k)$$

$$\left(\sqrt{h^2+k^2}\right)^2=\left(\sqrt{(4-h)^2+k^2}\right)^2$$

$$h^2+k^2=16+h^2-8h+k^2\Rightarrow 8h=16\Rightarrow h=2$$

The radius of the required circle is $r=|CA|=\sqrt{4+k^2}$ and the center is C(2,k).

For the values of k, the perpendicular distance from the center (2,k) on the line 3x +4y +4 = 0 equals the radius of the circle:

$$\frac{3(2)+4(k)+4}{\sqrt{9+16}}=\sqrt{4+k^2}$$

$$\frac{4k+10}{5}=\sqrt{4+k^2}$$

$$4k+10=5\sqrt{4+k^2},\ \text{squaring both sides}$$

$$16k^2+100+80k=25(4+k^2)$$

$$-9k^2+80k=0\Rightarrow -k(9k-80)=0\Rightarrow k=0, k=80/9$$

The coordinates of the center are (2,0) and (2, 80/9) and the radii are

$$r=\sqrt{4+0}=2 \text{ and } r=\sqrt{4+\frac{80^2}{9^2}}=\sqrt{\frac{6724}{81}}=\frac{82}{9}.$$

The equations of the circles with the above centers and radii are the following:

$$(x-2)^2+y^2=4, \qquad (x-2)^2+\left(y-\frac{80}{9}\right)^2=\frac{6724}{81}$$

Exercise 7.1

1. In each case, find an equation of a circle, when the center and radius are the following:

a. (0,0), r = 4 b. (3,2), r = 1

c. (–4,–3), r = 4 d. (–a,–b), r = a + b

2. In each case, determine the equation of a circle using the given information:

a. C(0,0), tangent to the line $x=-5$

b. C(0,0), tangent to the line y = 6

c. C(6,– 6), circumference passes through the origin.

d. C(0,5), circumference passes through the point (5,0).

e. C(– 9,– 6), circumference passes through the point (– 20,8).

f. C(2,– 8), circumference passes through the point (– 10,– 6).

g. C(– 5,4), tangent to the x-axis.

h. C(5,3), tangent to the y-axis.

3. In each case, find the center C(-g,-f) and radius $r=\sqrt{g^2+f^2-c}$ of the following:

a. $x^2+y^2-8x-6y+9=0$ b. $4x^2+4y^2+16x-12y-7=0$

c. $x^2+y^2+4x-6y+13=0$ d. $x^2+y^2-x-8y+18=0$

4. In each case, determine whether the given equation represents a circle. If no, state why not, if it is, then state the coordinates of the center and the radius:

a. $x^2+y^2-8x-4y+16=0$ b. $x^2+y^2+8y+6x=0$

c. $x^2+y^2-y-2=0$ d. $x^2+y^2+5x=0$

e. $3x^2+3y^2+6x-6y=0$ f. $2x^2+2y^2-8x+12y+8=0$

g. $x^2+2y^2-2x-2y=0$ h. $3x^2+2y^2+3x+2y=0$

i. $x^2+y^2+25=0$ j. $x^2+y^2+16=0$

5. In each case, find an equation of a circle which passes through the three points:

a. $(-3,0),(5,4),(6,-3)$ b. $(7,-1),(5,3),(-4,6)$

c. $(1,2),(3,-4),(5,-6)$ d. $(-3,4),(-2,0),(1,5)$

6. In each case, find an equation of a circle which

a. contains the point $(2,6),(6,4)$ and has its center on the line $3x+2y-1=0$.

b. contains the point $(4,1),(6,5)$ and has its center on the line $4x+y-16=0$.

c. contains the points $(1,-2),(4,3)$ and has its center on the line $3x+4y-7=0$.

d. contains the point $(0,3),(4,1)$ and has its center on the x-axis.

7. Find an equation of a circle which passes through the points

a. $(0,0),(0,3)$ and the line $4x-5y=0$ is tangent to it at (0,0).

b. $(0,-1),(3,0)$ and the line $3x+y=9$ is tangent to it at (3,0).

c. $(0,1),(3,-1)$ and the line $2x+2y-2=0$ is tangent to it at (0,1).

d. $(0,4),(2,6)$ and the line $x+y-4=0$ is tangent to it at (0,4).

8. Find an equation of a circle which passes through the point (– 3,0) and is concentric (one circle having a common center with another circle) to circle $x^2+y^2-3x-4y-10=0$.

9. Find an equation of a circle that is concentric to circle

a. $2x^2+2y^2+16x-7y=0$ and is tangent to the y-axis.

b. $x^2+y^2-8x+4=0$ and is tangent to the line $x+2y+6=0$.

c. $x^2+y^2+6x-10y+33=0$ and is touching the x-axis.

10 In each case, find an equation of a circle which passes through the origin, whose intercepts on the coordinate axes are:

a. 3 and 4 b. 2 and 4

7.3 Tangents and Normal

If a secant PQ of a circle is moved upward about one of its points of intersection P, then the second point of intersection Q is moving gradually along the curve that tends to coincide with P. The limiting position PT of PQ is then called the **tangent** to the circle at the point P.

The point of the circle at which a tangent meets the circle is called **point of contact** of the tangent.

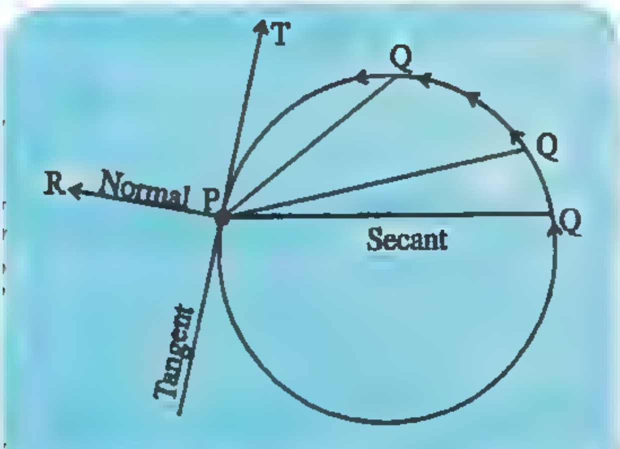


Fig. 7.5

The **normal** at a contact point P to a circle (or conic) is the straight line PR perpendicular to the tangent PT to the circle (or conic) at that point P.

i) *To find a Condition when a line intersects the circle*

The circle and the line are

$$x^2 + y^2 = a^2 \quad (19)$$

$$y = mx + c \quad (20)$$

which develops a system of two equations:

$$x^2 + y^2 = a^2$$
$$y = mx + c \qquad (21)$$

The solution set $\{(x, y)\}$ of the system of equations (21) exists only, if the curves of the system (21) are intersecting. That set of points of intersection $\{(x, y)\}$ is the solution set, can be found by solving the nonlinear system (21) simultaneously.

The line (20) is used in a circle (19) to obtain the quadratic equation in x:

$$x^2 + (mx+c)^2 = a^2$$
$$x^2(1+m^2) + 2mcx + (c^2 - a^2) = 0 \qquad (22)$$

The equation (22) being a quadratic equation in x, gives a set of two values x_1 and x_2 of x which will be used in a line (20) to obtain a set of two y values y_1 and y_2.

The solution set $\{(x_1, y_1), (x_2, y_2)\}$ of the system (21) is of course a set of points of intersection of the line and circle.

The points of intersection of the system (21) are real, coincident or imaginary, according as the roots of the quadratic equation (22) are real, coincident or imaginary or according as the discriminate of the quadratic equation (22):

$$disc = 4m^2c^2 - 4\left[(1+m^2)(c^2-a^2)\right] > 0, \quad \text{real and distant}$$
$$disc = 4m^2c^2 - 4\left[(1+m^2)(c^2-a^2)\right] = 0, \quad \text{coincident}$$
$$disc = 4m^2c^2 - 4\left[(1+m^2)(c^2-a^2)\right] < 0, \quad \text{imaginary}$$

Example 7.3.1: [Point of Intersection]: Find the points of intersection of the line $3x - 4y + 20 = 0$ and the circle $x^2 + y^2 = 25$.

Solution: The equations of the line and circle are:

$$3x - 4y + 20 = 0$$
$$y = \frac{3}{4}x + 5 \qquad (23)$$
$$x^2 + y^2 = 25 \qquad (24)$$

The line (23) is used in a circle (24) to obtain the x-coordinates of the points of intersection:

$$x^2+\left(\frac{3}{4}x+5\right)^2=25$$
$$x^2+\frac{9}{16}x^2+25+\frac{30}{4}x=25$$
$$x^2+\frac{9}{16}x^2+\frac{30}{4}x=0$$
$$25x^2+120x=0$$
$$x=0,-24/5$$

The x-coordinates x=0, -24/5 are used in the line (23) to obtain the y-coordinates:

$$x=0 \text{ gives } y=5$$
$$x=-24/5 \text{ gives } y=\frac{3}{4}\left(\frac{-24}{5}\right)+5=\frac{-72+100}{20}=\frac{28}{20}=\frac{7}{5}$$

Thus, the points of intersection (0, 5) and (– 24/5,7/5) are real and distant.

Example 7.3.2: [point of Intersection]: Find the points of intersection of the line $x+3y-5=0$ and the circle $x^2+y^2-2x+4y-5=0$.

Solution: The equations of the line and circle are:

$$x+3y-5=0$$
$$x=5-3y \quad (25)$$
$$x^2+y^2-2x+4y-5=0 \quad (26)$$

The line (25) is used in a circle (26) to obtain the y-coordinates of the points of intersection:

$$(5-3y)^2+y^2-2(5-3y)+4y-5=0$$
$$y^2-2y+1=0$$
$$y=1,1$$

The y-coordinates y =1,1 are used in the line (25) to obtain the x-coordinates:

$y = 1, 1$ give $x = 2, 2$

Thus, the points of intersection (2,1) and (2,1) are coincident and the line $x+3y-5=5$ touches the circle $x^2+y^2-2x+4y-5=0$ at a point (2,1).

ii) To find a condition when a line touches the circle

Let AB be the straight line y=mx+c that intersects the circle $x^2+y^2=a^2$ at points P and Q respectively.

Join OP and put it by OP = a, which is the radius of a given circle. Draw OM perpendicular on PQ. If OM is perpendicular to PQ, then, the perpendicular distance OM from O(0,0) on a secant line $mx-y+c=0$ (line PQ) is:

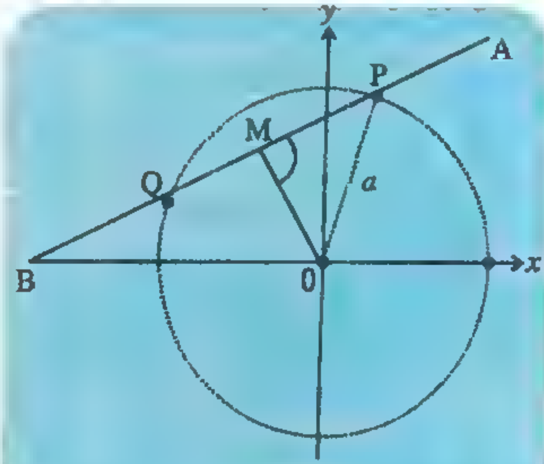


Fig. 7.6

$$OM = \frac{m(0)-(0)+c}{\sqrt{m^2+1}} = \frac{c}{\sqrt{1+m^2}}$$

From the right-angled triangle OMP, it is known that:

$$|OP|^2 = |OM|^2 + |MP|^2$$
$$|MP|^2 = |OP|^2 - |OM|^2$$
$$= a^2 - \frac{c^2}{1+m^2} = \frac{a^2(1+m^2)-c^2}{1+m^2}$$
$$|MP| = \sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$$

The secant line PQ is 2 times of MP, and the length of the intercept PQ is therefore:

$$|PQ| = 2|MP| = 2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}} \qquad (27)$$

Condition of Tangency: The line y=mx + c touches the circle $x^2+y^2=a^2$, if the length of the intercept PQ is zero:

$$2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}=0$$

$$2\frac{\sqrt{a^2(1+m^2)-c^2}}{\sqrt{1+m^2}}=0.$$

$$\sqrt{a^2(1+m^2)-c^2}=0, \text{ squaring both sides}$$

$$a^2(1+m^2)-c^2=0 \Rightarrow c^2=a^2(1+m^2) \Rightarrow c=\pm a\sqrt{1+m^2} \qquad (28)$$

The equation (28) is the required condition at which the line y = mx + c touches the circle $x^2+y^2=a^2$.

Example 7.3.3: [Condition of Tangency]: Find the length of the chord joining the points P and Q on the line $\frac{x}{a}+\frac{y}{b}=1$ which cuts the circle $x^2+y^2=r^2$. Show that if the line touches the circle, then $a^2+b^2=r^2$.

Solution: The slope of a given line is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\frac{y}{b}=-\frac{x}{a}+1$$

$$y=-\frac{b}{a}x+b,\ m=-b/a$$

If PQ is the chord of a circle $x^2+y^2=r^2$, and PQ is 2 times of MP, then the length of the chord PQ through result (27) is:

$$|PQ|=2|MP|$$

$$=2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$$

$$=2\sqrt{\frac{r^2\left[1+(-b/a)^2\right]-b^2}{1+(-b/a)^2}}, \quad a^2=r^2,\quad m=-b/a,\quad c=b$$

If the given line touches the circle $x^2+y^2=r^2$, then, the length of the chord PQ is going to be zero:

$$2\sqrt{\frac{r^2\left[1+(-b/a)^2\right]-b^2}{1+(-b/a)^2}}=0$$

$$4\frac{r^2\left[1+(-b/a)^2\right]-b^2}{1+(-b/a)^2}=0, \quad \text{squaring both sides}$$

$$r^2\left[1+(-b/a)^2\right]-b^2=0$$

$$b^2=r^2\left[1+\frac{b^2}{a^2}\right]$$

$$b^2=r^2\frac{a^2+b^2}{a^2}$$

$$\frac{a^2b^2}{a^2+b^2}=r^2$$

$$r^{-2}=\frac{a^2+b^2}{a^2b^2}=\frac{1}{a^2}+\frac{1}{b^2}=a^{-2}+b^{-2}$$

Example 7.3.4: [Condition of Tangency]: Find the coordinates of the middle point of the chord which the circle $x^2+y^2+4x-2y-3=0$ cuts off on the line $x-y+2=0$.

Solution: The centre of the given circle is $C(-g,-f)=C(-2,1)$ and the line $x-y+2=0$ (line AB) intersects the circle at points P and Q and $M(x_1,y_1)$ is the middle point of the chord PQ. Join C and M that develops a line CM perpendicular to chord PQ.

If M lies on line AB, then, the line equation $x-y+2=0$ becomes:

$$x_1-y_1+2=0 \quad (29)$$

The slopes of the lines (29) and CM are respectively:

$$m_2=1, \text{ coefficient of } x$$

$$m_1=\frac{y_1-1}{x_1+2}, \text{ slop of } CM$$

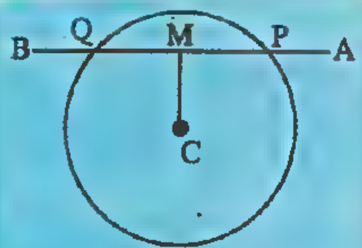


Fig. 7.7

If C M is perpendicular to AB, then the product of their slopes equals –1:

$$\left(\frac{y_1-1}{x_1+2}\right)(1)=-1 \Rightarrow y_1-1=-x_1-2 \Rightarrow x_1+y_1+1=0 \quad (30)$$

The equations (29) and (30) are solved to obtain the coordinates of the middle point M:

$$\left.\begin{array}{l} x_1-y_1+2=0 \\ x_1+y_1+1=0 \end{array}\right\} x_1=-3/2,\ y_1=1/2$$

Thus, the coordinates of the middle point is M(– 3/2, 1/2).

iii) The Equation of a tangent to a circle in slope form

If m is the slope of the tangent line to the circle

$$x^2+y^2=a^2 \quad (31)$$

then the equation of that tangent line is of the form

$$y=mx+c \quad (32)$$

Here c is to be calculated from the fact that the line (32) is tangent to the circle (31). The line (32) is used in circle (31) to obtain the quadratic equation in x:

$$x^2+(mx+c)^2=a^2, \quad y=mx+c$$
$$x^2\left(1+m^2\right)+2mcx+\left(c^2-a^2\right)=0 \qquad (33)$$

If the line (32) touches the circle (31), then the quadratic equation (33) has coincident roots for which the discriminant of the quadratic equation (33) equals zero:

$$4m^2c^2-4\left(1+m^2\right)\left(c^2-a^2\right)=0$$
$$4m^2c^2=4\left(1+m^2\right)\left(c^2-a^2\right)$$
$$m^2c^2=m^2c^2+c^2-a^2-a^2m^2 \qquad (34)$$
$$-c^2=-a^2\left(1+m^2\right)$$
$$c=\pm a\sqrt{1+m^2}$$

Equation (34) is the **condition of tangency**. The value of c from equation (34) is used in the line (32) to obtain the required equation of the tangent:

$$y=mx+c=mx\pm a\sqrt{1+m^2} \qquad (35)$$

This develops the following facts:

- The equation of any tangent to the circle $x^2+y^2=a^2$ in the slope form is:

$$y=mx\pm a\sqrt{1+m^2} \qquad (36)$$

- **Condition of Tangency:** The line $y=mx+c$ should touch the circle $x^2+y^2=a^2$ under condition:

$$c=\pm a\sqrt{1+m^2} \qquad (37)$$

- The interpretation of result (35) is that the line $lx+my+n=0$ should touch the circle $x^2+y^2=a^2$ if

$$n=\pm a\sqrt{l+m^2} \qquad (38)$$

- The interpretation of result (35) that the line $lx+my+n=0$ should touch the circle $x^2+y^2+2gx+2fy+c=0$ if

$$\left(c-f^2\right)l^2+2fglm+\left(c-g^2\right)m^2-2n\left(gl+fm\right)+n^2=0 \qquad (39)$$

- **Point of Contact:** Let $y=mx\pm a\sqrt{1+m^2}$ be a tangent to a circle (31) at a point $(x_1,\ y_1)$, if the circle equation (31) is identical to $xx_1+yy_1=a^2$, then the coefficients of like terms of $y=mx\pm a\sqrt{1+m^2}$ and $xx_1+yy_1=a^2 \Rightarrow yy_1=-xx_1+a^2$ are compared to obtain the point of contact:

$$\frac{x_1}{-m}=\frac{y_1}{1}=\frac{a^2}{a\sqrt{1+m^2}}$$

$$\frac{x_1}{-m}=\frac{a}{\sqrt{1+m^2}}$$

$$x_1=\frac{-am}{\sqrt{1+m^2}}$$

$$\frac{y_1}{1}=\frac{a}{\sqrt{1+m^2}}$$

$$y_1=\frac{a}{\sqrt{1+m^2}}$$

Thus, the point of contact is $(x_1,y_1)=\left(\frac{-am}{\sqrt{1+m^2}},\frac{a}{\sqrt{1+m^2}}\right)$. (40)

Example 7.3.5: [Tangency Condition and Point of Contact]: For what value of c, the line $x+y+c=0$ will touch the circle $x^2+y^2=64$? Use that value of c to find the tangent that should touch the given circle. Find also the contact point.

Solution: The slope of the line x+y+c=0 is $m=-1$. The value of c at which the line $x+y+c=0$ will touch the given circle $x^2+y^2=64$ is:

$$c=\pm a\sqrt{1+m^2},\ result\,(37)$$

$$=\pm 8\sqrt{1+(-1)^2}=8\sqrt{2},\quad a=8,\ m=-1$$

The required tangent line that should touch the given circle is:

$$y=mx\pm a\sqrt{1+m^2}=-x\pm 8\sqrt{2},\ result\,(36)$$

The point of contact through result (40) is:

$$(x_1, y_1) = \left(\frac{-am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right) = \left(\frac{8}{\sqrt{2}}, \frac{8}{\sqrt{2}}\right)$$

iv) *The equations of tangent and normal to a circle at a point*

The equation of a circle is:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (41)$$

If $A(x_1, y_1)$ is a point lying on the circle (41), then the circle (41) becomes:

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad (42)$$

If r_1 and r_2 are the position vectors of A and the center C(−g, −f) of the circle relative to origin

$$r_1 = (x_1, y_1) = x_1 i + y_1 j, \quad r_2 = (-g, -f) = -gi - fj$$

then, from the figure (7.8):

$$OC + CA = OA$$

$$CA = OA - OC = r_1 - r_2 = (x_1 + g, y_1 + f) = (x_1 + g)i + (y_1 + f)j$$

Let $P(x, y)$ be any point on the tangent line AT, whose position vector is $OP = (x, y)$ which gives:

$$OA + AP = OP$$

$$AP = OP - OA$$

$$= r - r_1 = (x, y) - (x_1, y_1) = (x - x_1, y - y_1) = (x - x_1)i + (y - y_1)j$$

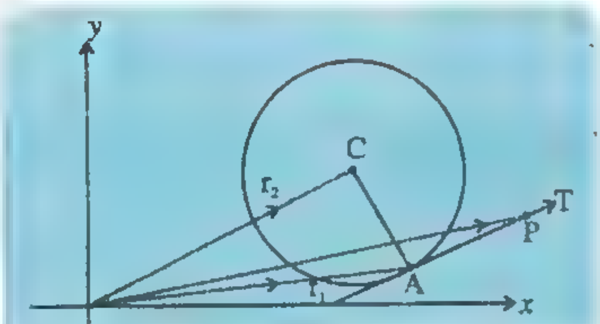


Fig. 7.8

The equation of tangent to the circle (41) is obtained if AP is perpendicular to CA for which the dot product in between the vectors AP and AC equals zero:

$$AP.CA=0$$
$$(x_1+g, y_1+f).(x-x_1, y-y_1)=0$$
$$(x_1+g)(x-x_1)+(y_1+f)(y-y_1)=0$$
$$xx_1+yy_1+gx+fy-(x_1^2+y_1^2+gx_1+fy_1)=0$$
$$xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1$$
$$=-gx_1-fy_1-c, \qquad res(42)$$
$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0 \qquad (43)$$

The tangent equation to the circle $x^2+y^2=a^2$ at a point $A(x_1, y_1)$ through result (43) is:

$$xx_1+yy_1=a^2 \qquad (44)$$

The procedure for the normal equation at a point (x_1, y_1) on the circle $x^2+y^2+2gx+2f\,y+c=0$ is as under:

If C(– g, – f) is the center of the circle and $A(x_1, y_1)$ is a contact point, then the slope $\frac{y_1+f}{x_1+g}$ of the required normal line develops the normal line CA at $A(x_1, y_1)$:

$$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1)$$
$$(y-y_1)(g+x_1)=(f+y_1)(x-x_1)$$
$$x(y_1+f)-y(x_1+g)+(gy_1-fx_1)=0 \qquad (45)$$

The normal equation to the circle $x^2+y^2=a^2$ at a point $A(x_1, y_1)$ through result (45) is:

$$xy_1-yx_1=0 \qquad (46)$$

Example 7.3.6: [Tangent and Normal]: Find the equations of the tangent and normal to the circle $x^2+y^2=25$ at a point (3,4).

Solution: Result (44) is used to obtain the tangent equation to the given circle:

$$xx_1+yy_1=a^2$$

$$3x+4y=25,\quad a^2=25,\ (x_1,y_1)=(3,4)$$

Result (46) is used to obtain the normal equation to the given circle:

$$xy_1-yx_1=0$$

$$4x-3y=0,\quad a^2=25,\ (x_1,y_1)=(3,4)$$

Example 7.3.7:[Tangent and Normal Equations]: Find the equations of the tangent and normal to the circle $x^2+y^2-2x+4y+3=0$ at a point (2,- 3).

Solution: Result (43) is used to obtain the tangent line to the given circle:

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

$$2x-3y+(-1)(x+2)+(2)(y-3)+3=0,\ 2g=-2,\ 2f=4,\ c=3$$

$$2x-3y-x-2+2y-6+3=0,\ (x_1,y_1)=(2,-3)$$

$$x-y-5=0$$

Result (45) is used to obtain the normal line to the given circle:

$$x(y_1+f)-y(x_1+g)+(gy_1-fx_1)=0$$

$$x(-3+2)-y(2-1)+(3-4)=0,\ 2g=-2,\quad 2f=4,\quad c=3$$

$$-x-y-1=0$$

$$x+y+1=0$$

v) *Two tangents drawn to a circle from an external point are equal in length*

If $y=mx\pm a\sqrt{1+m^2}$ is any tangent to the circle $x^2+y^2=a^2$, then the tangent line that passes through the point (x_1,y_1) is

$$y_1 = mx_1 + a\sqrt{1+m^2}, \; result\,(35)$$

$$y_1 - mx_1 = a\sqrt{1+m^2} \qquad (47)$$

Squaring (47)

$$(y_1 - mx_1)^2 = a^2(1+m^2)$$

that gives the quadratic equation in m:

$$m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - a^2) = 0 \qquad (48)$$

This quadratic equation (48) gives two values of m those two values of m represent the slopes of the required two tangents on the given circle.

The tangents are real and different, real and coincident or imaginary according as the discriminant of the quadratic equation (48):

$$disc = 4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) > 0, \quad \textit{real and different}$$
$$disc = 4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) = 0, \quad \textit{real and coincident}$$
$$disc = 4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) < 0, \quad \textit{imaginary}$$

or according as

$$x_1^2 + y_1^2 - a^2 > 0, \quad \therefore \textit{real and different}$$
$$x_1^2 + y_1^2 - a^2 = 0, \quad \therefore \textit{real and coincident}$$
$$x_1^2 + y_1^2 - a^2 < 0, \quad \therefore \textit{imaginary}$$

or according as the point $P(x_1, y_1)$ lies outside, on, or inside the circle $x^2 + y^2 = a^2$.

Example 7.3.8: [Two Tangents]: Find the equations of the tangents drawn from the point (6,4) to the circle $x^2 + y^2 = 16$.

Solution: If $y = mx \pm c = mx \pm a\sqrt{1+m^2}$ are any tangents to the circle $x^2 + y^2 = 16$, then the number of tangents through result (48)

$$m^2\left(x_1^2-a^2\right)-2mx_1y_1+\left(y_1^2-a^2\right)=0$$

$$m^2(36-16)-2m(6)(4)+(16-16)=0,\ (x_1,y_1)=(6,4),\quad a^2=16$$

$$20m^2-48m=0$$

$$m(20m-48)=0$$

$$m=0,12/5$$

can be found by putting m = 0 and m=12/5 in $y=mx\pm a\sqrt{1+m^2}$:

a: $\quad y=mx\pm a\sqrt{1+m^2}=(0)x\pm4\left(\sqrt{1+0}\right)=\pm4=4,\quad a=4,\quad m=0$

b. $\quad y=\left(\dfrac{12}{5}\right)x\pm4\left(\sqrt{1+\dfrac{144}{25}}\right)$

$$=\frac{12}{5}x\pm4\left(\frac{13}{5}\right)=\frac{12}{5}x\pm\frac{52}{5}=\frac{12}{5}x-\frac{52}{5},\text{ choose negative sign}$$

vi) The length of tangent to a circle from a given external point

The procedure for finding the length of the tangent drawn from the external point $P(x_1,y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$ is as under:

Let $P(x_1,y_1)$ be the given external point and PT be one of the two tangents drawn from point P to the circle

$$x^2+y^2+2gx+2fy+c=0 \qquad (49)$$

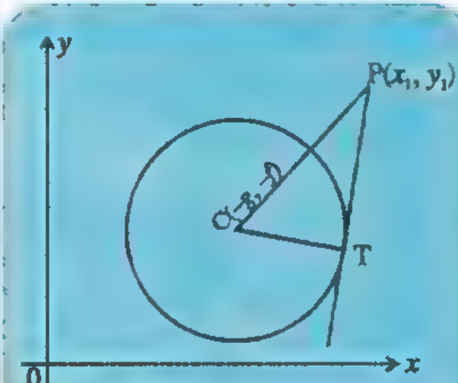


Fig. 7.9

Join CP and CT. C (−g, −f) is the center of the circle (49) and $CT=\sqrt{g^2+f^2-c}$ is the radius of the circle (49).

From the right-angled triangle PTC, the length of the tangent PT drawn from point P to the given circle is:

$$|PC|^2=|CT|^2+|PT|^2$$

$$|PT|^2=|PC|^2-|CT|^2$$

$$=(x_1+g)^2+(y_1+f)^2-(g^2+f^2-c)$$

$$=x_1^2+y_1^2+2gx_1+2f\,y_1+c$$

$$|PT|=\sqrt{x_1^2+y_1^2+2gx_1+2f\,y_1+c} \qquad (50)$$

It is important to note that:

- the length of the tangent drawn from the point $P(x_1,y_1)$ to the circle $x^2+y^2=a^2$ is:

$$|PT|=\sqrt{x_1^2+y_1^2-a^2} \qquad (51)$$

- the lengths of the two tangents drawn from the point $P(x_1,y_1)$ on the given circle are equal.

Example 7.3.9: [Length of the Tangent]: Find the length of the tangent drawn from the point P(3,4) on the circles

a. $x^2+y^2=9$

b.. $x^2+y^2-2x-y-3=0$

Solution:

a. If PT is the tangent drawn from the point P(3,4) on the given circle, then, the length of the tangent PT on the given circle through result (51) is:

$$|PT|=\sqrt{x_1^2+y_1^2-9}=\sqrt{9+16-9}=\sqrt{16}=4,\ \ (x_1,y_1)=(3,4),\ a^2=9$$

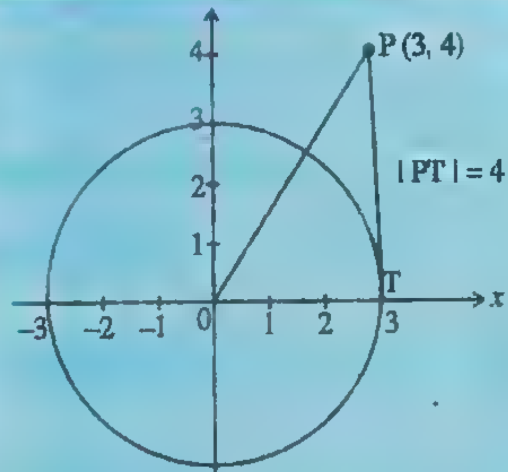


Fig. 7.10

b. If PT is the tangent drawn from the point P(3,4) on the given circle, then, the length of the tangent PT on the given circle through result (50) is:

$$|PT| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2f\,y_1 + c}, 2g = -2,\ 2f = -1,\ c = -3, (x_1, y_1) = (3,4)$$

$$= \sqrt{9 + 16 - 2(3) - (4) - 3} = \sqrt{12}$$

Example 7.3.10: [Length of the Tangent]: Find the length of the two tangents drawn from the point P(2, 3) on the circle $x^2 + y^2 - 2x + 4y - 18 = 0$.

Solution: If PT and PM are the two tangents drawn from the point P(2, 3) on the given circle, then, the length of the tangent PT from point P(2, 3) equals the length of the tangent MP from the point P(2, 3):

$$|PT| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2f\,y_1 + c}$$

$$= \sqrt{4 + 9 - 2(2) + 2(2)(3) - 18},\ 2g = -2,\ 2f = 4, c = -18, (x_1, y_1) = (2, 3)$$

$$= \sqrt{3} = |PM|$$

Example 7.3.11: [Two Tangents]: Prove that the lines x = 7 and y = 8 touch the circle $x^2 + y^2 - 4x - 6y - 12 = 0$. Find also the contact points.

Solution: Before solving the problem, the idea is that "any line is tangent to the circle, if the perpendicular distance from the center of the circle on the given line equals the radius of the circle".

The center and radius of the circle $x^2+y^2-4x-6y-12=0$ are respectively:

$$C(-g,-f)=C(2,3),\quad c=-12,\quad r=\sqrt{g^2+f^2-c}=\sqrt{4+9+12}=5$$

The perpendicular distance d from C(2,3) on the line x–7 = 0

$$d=\left|\frac{1(2)+0(3)-7}{\sqrt{1+0}}\right|=|-5|=5$$

which is equal to the radius of the circle r = 5. Thus, the line x = 7 touches the given circle. To find the point of the contact, put x = 7 in the given circle equation to obtain the y-coordinate:

$$x^2+y^2-4x-6y-12=0$$
$$49+y^2-28-6y-12=0,\qquad x=7$$
$$y^2-6y+9=0$$
$$(y-3)^2=0$$
$$y-3=0$$
$$y=3$$

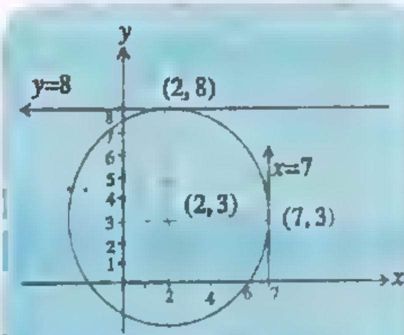


Fig. 7.11

Thus, the contact point in between the line $x=7$ and the circle $x^2+y^2-4x-6y-12=0$ is (7,3).

It is understood that the line y=8 also touches the given circle $x^2+y^2-4x-6y-12=0$ and the contact point in between these two curves is (2,8).

Example 7.3.12: [Two Tangents]: Find the equations of two tangents to the circle $x^2+y^2=25$ which are parallel to the line $3x+4y-7=0$.

Solution: Before solving the problem, the idea is that any line parallel to the given line $3x+4y-7=0$ is:

$$3x+4y+t=0 \qquad (52)$$

If a line (52) is tangent to the circle $x^2+y^2=25$, then the perpendicular distance d form the center C(0,0) to the given line (52)

$$d=\left|\frac{3(0)+4(0)+t}{\sqrt{9+16}}\right|=\left|\frac{t}{5}\right|=\pm\frac{t}{5}$$

equals the radius of the circle $\pm\frac{t}{5}=5$ that gives $t=+25,\ -25$.

Thus, the two tangent equations to the given circle for t = 25 and t = – 25 are the following:

$3x+4y+25=0, \qquad t=25$

$3x+4y-25=0, \qquad t=-25$

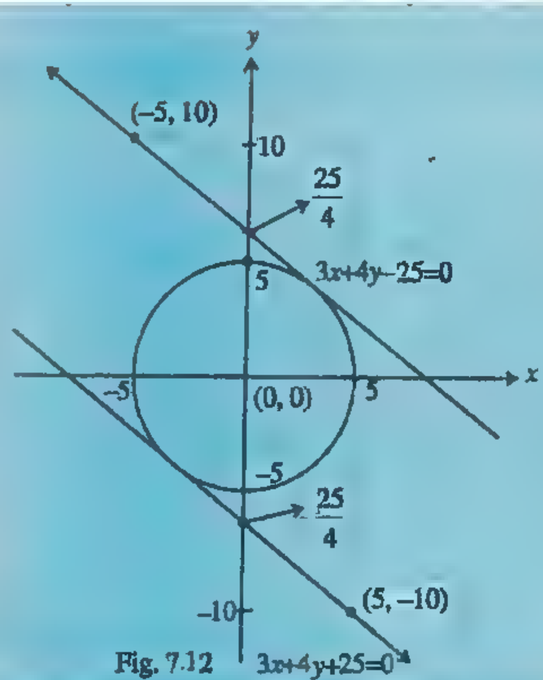


Fig. 7.12

Exercise 7.2

1. In each case, find the tangent and normal equations

a. at a point (1,2) to the circle $x^2+y^2=5$.

b. at a point (– 1,3) to the circle $x^2+y^2+6x-y-1=0$.

2. In each case, find the tangent and normal equations

a. at a point $(2\cos 45°, 2\sin 45°)$ to the circle $x^2+y^2=4$.

b. at a point $(\cos 30°, \sin 30°)$ to the circle $x^2+y^2=1$.

3. Find the condition that

a. the line $x+y+n=0$ touches the circle $x^2+y^2=9$.

b. the line $2x+2y+n=0$ touches the circle $x^2+y^2=81$.

4. For what value of n,

a. the line $x+y+n=0$ touches the circle $x^2+y^2=9$?

b. the line $2x+2y+n=0$ touches the circle $x^2+y^2=81$?

5. For what value of c

a. a. the line $y=mx+c$ touches the circle $x^2+y^2=a^2$??

a. the line $y=-x+c$ touches the circle $x^2+y^2=9$?

6. Find the condition at which the line $lx+my+n=0$ touches the circle $x^2+y^2+2gx+2fy+c=0$.

7. For what value of n,

a. the line $3x+4y+n=0$ touches the circle $x^2+y^2-4x-6y-12=0$?

b. the line $x-2y+n=0$ touches the circle $x^2+y^2+3x+6y-5=0$?

8. Find the locus of the point P, if the length of the tangent line from the point P to the circle $x^2+y^2=9$ is equal to the perpendicular distance form P to the line $3x+4y+3=0$.

9. The length of the tangent from (f, g) to the circle $x^2+y^2=6$ is twice the length of the tangent to the circle $x^2+y^2+3x+3y=0$. Prove that $f^2+g^2+4f+4g+2=0$.

10. Find the equations of the tangents to the circle $x^2+y^2=25$ which are parallel to the straight line $3x+4y+3=0$.

a. Prove that the lines $x = 8$ and $y = 7$ touch the circle $x^2 + y^2 - 6x - 4y - 12 = 0$ Find also the contact points.

b Prove that the lines $x+y-1=0$ and $x-y+1=0$ touch the circle $x^2 + y^2 - 4x - 2y + 3 = 0$. Find also the contact points.

12. Find the equations of the tangents to the circle $x^2 + y^2 = 2$, which make an angle of $45°$ with the x-axis.

13 Find the equations of the tangents drawn from the point $P(4,3)$ to the circle $x^2 + y^2 = 9$.

There are some properties of a circle that are listed as under.

- ***Perpendicular from the center of a circle on a chord bisects the chord***

Let the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (53)$$

and PQ be any chord of a circle, whose end points are $P(x_1, y_1)$ and $Q(x_2, y_2)$ respectively.

If PQ is a chord of the circle, then P and Q are the points lying on the circle:

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$
$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$$

The subtraction of these two circle equations gives the slope of the chord PQ

$$\left(x_2^2 - x_1^2\right) + \left(y_2^2 - y_1^2\right) + 2g\left(x_2 - x_1\right) + 2f\left(y_2 - y_1\right) = 0$$
$$\left(x_2^2 - x_1^2\right) + 2g\left(x_2 - x_1\right) + \left(y_2^2 - y_1^2\right) + 2f\left(y_2 - y_1\right) = 0$$
$$\left(x_2 - x_1\right)\left(x_1 + x_2 + 2g\right) + \left(y_2 - y_1\right)\left(y_1 + y_2 + 2f\right) = 0$$

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_1 + x_2 + 2g}{y_1 + y_2 + 2f} = m_1, \text{ say.} \qquad (54)$$

If the centre of the circle is C(– g,– f) and the midpoint of the chord PQ is $D\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, then the slope of the perpendicular line CD is:

$$m_2 = \frac{\frac{y_1+y_2}{2}+f}{\frac{x_1+x_2}{2}+g} = \frac{y_1+y_2+2f}{x_1+x_2+2g} \qquad (55)$$

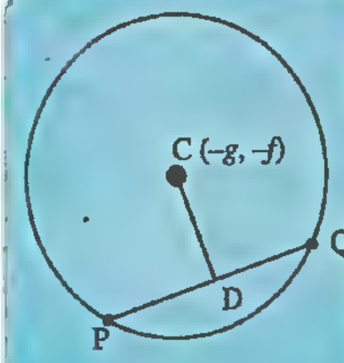


Fig. 7.13

From the Fig. 7.13 the chord PQ and the line CD are perpendicular if and only if the product of their slopes equals –1:

$$m_1 m_1 = -\frac{x_1+x_2+2g}{y_1+y_2+2f}\cdot\frac{y_1+y_2+2f}{x_1+x_2+2g} = -1$$

Thus, CD is bisector of the chord PQ.

It is important to note that:

- the perpendicular bisector of any chord PQ of a circle passes through the centre of the circle. This is our **second property**.
- the line joining the two points of the circle passes through the centre of the circle is called the diameter of the circle. This diameter acts as the perpendicular bisector to the chord PQ, if the diameter of a circle bisects the chord PQ. This is our **third property**. The proof is similar to property first, but the graphical view is shown in the Fig. 7.14.

Example 7.4.1:[Perpendicular Bisector]: If $A(-3,4)$ and $B(1,5)$ are the end points of the chord AB of the circle $x^2+y^2+x-5y-2=0$, then show that

a. the line from the centre of the circle is perpendicular to AB, also bisects the chord AB.

b. the line from the centre of the circle to the midpoint of the chord AB is perpendicular to the chord AB.

c. the perpendicular bisector CD of the chord AB passes though the centre of the given circle.

Solution: The equation of the circle is:

$$x^2+y^2+x-5y-2=0$$

If the center of the circle is $C(-1/2,5/2)$ and the midpoint of the chord AB is $D(-1,9/2)$, then the slopes of the chord AB and the perpendicular line CD are respectively:

$$m_1=\frac{5-4}{1+3}=\frac{1}{4},\ \text{slope of AB},\quad m_2=\frac{\frac{9}{2}-\frac{5}{2}}{-1+\frac{1}{2}}=-4,\ \text{slope of CD}$$

The chord AB and the line CD are perpendicular if and only if the product of their slopes equals –1:

$$m_1m_1=\frac{1}{4}.(-4)=-1$$

Thus, CD is perpendicular bisector of the chord AB. This result is automatically valid for parts b and c.

- ***Congruent chords of a circle are equidistant from its center***

If the perpendicular distances form the center of a circle to its two chords are equal, then the chords are congruent.

Let the circle equation with center C(– g, – f) is:

$$x^2+y^2+2gx+2fy+c=0 \qquad (56)$$

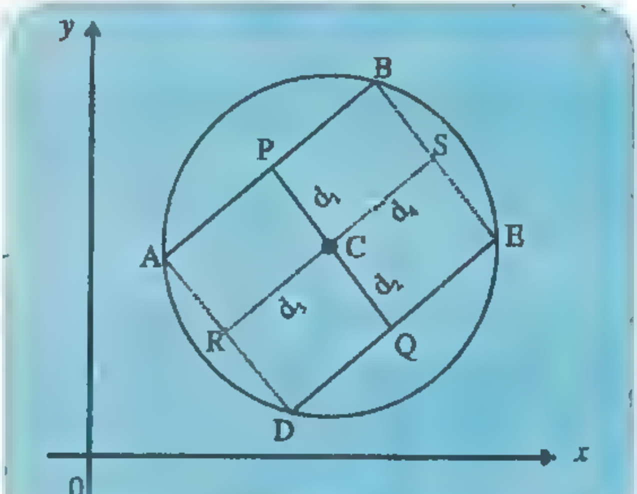


Fig. 7.14

If AB and DE are the two chords of the circle (56), then the coordinates of the end points of the chord AB and DE are respectively:

$$A(x_1, y_1), D(x_2, y_2), B(x_3, y_3), E(x_4, y_4)$$

From the Fig. 7.14, it is clear that the perpendicular distance $d_1 = CP$ from the center C on the chord AB equals the perpendicular distance $d_2 = CQ$ from C on the chord DE, if and only if the chords AB and DE are with equal lengths:

$$|AB| = |DE|$$

Thus, the chords AB and DE are equidistant from C on the circle (56) if and only if

$$d_1 = d_2 \qquad (57)$$

In similar manner, the chords AD (join A to D) and BE (join B to E) are congruent chords, if the perpendicular distance $d_3 = CR$ from C on the chord AD equals the perpendicular $d_4 = CS$ from C on the chord BE:

$$|AD| = |BE| \qquad (58)$$

Example 7.4.2:[Congruent Chords]: Show that the chords AB and DE are equidistant from the centre C(0,0) of the circle $x^2 + y^2 = 4$. The coordinates of the end points of the two chords are A(0,2), B(−2,0), D(0,−2) and E(2,0).

Solution: The circle equation with center C(0,0) is:

$$x^2 + y^2 = 4 \qquad (59)$$

If AB and DE are the two chords of the circle (59), whose coordinates are respectively:

$$A(0,2), \quad B(-2,0), \quad D(0,-2), \quad E(2,0)$$

From the Fig. 7.15, it is clear that the chords AB and DE are with equal length:

$$AB = (-2-0, 0-2), \quad |AB| = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$$

$$DE = (2-0, 0+2), \quad |DE| = \sqrt{(2)^2 + (2)^2} = 2\sqrt{2}$$

Thus, the two chords AB and DE are equal. For equidistant, the procedure is as under:

The equations of the chords AB and DE (through two-point form of the line) are respectively:

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-2}{x-0}=\frac{0-2}{-2-0},\quad A(x_1,y_1)=A(0,2),\ B(x_2,y_2)=B(-2,0)$$

$$\frac{y-2}{x}=1 \Rightarrow x-y+2=0$$

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y+2}{x-0}=\frac{0+2}{2-0},\quad D(x_1,y_1)=D(0,-2),\ E(x_2,y_2)=E(2,0)$$

$$\frac{y+2}{x}=1 \Rightarrow x-y-2=0$$

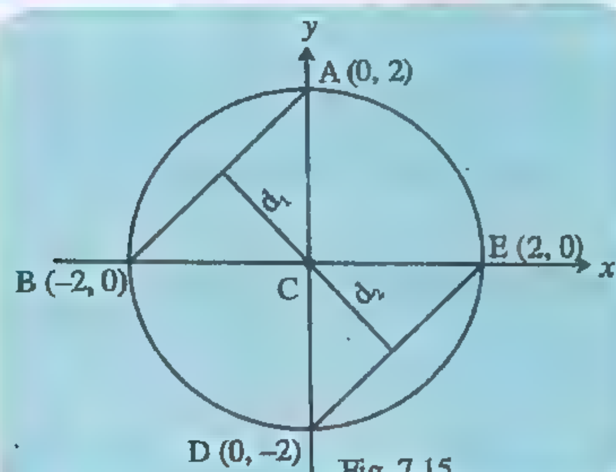


Fig. 7.15

The perpendicular distance d_1 from C(0,0) on the chord AB is:

$$d_1=\left|\frac{0-0+2}{\sqrt{1+1}}\right|=\frac{2}{\sqrt{2}}$$

The perpendicular distance d_2 from C(0,0) on the chord DE is:

$$d_2 = \left|\frac{0-0-2}{\sqrt{1+1}}\right| = \frac{2}{\sqrt{2}}$$

The perpendicular distance d_1 from C(0,0) on the chord AB is equal to the perpendicular distance d_2 form C(0,0) on the chord DE:

$$d_1 = d_2 = 2/\sqrt{2}$$

Thus, the chords AB and DE are equidistant from the center C(0,0) of the circle (59).

- ***Measure of the central angle of a minor arc is double the measure of the angle subtended by the corresponding major arc***

Let the circle be

$$x^2 + y^2 = a^2 \qquad (60)$$

The arc BC is the minor arc of the circle (60), whose coordinates are $B(-x_1, -y_1)$ and $C(x_1, -y_1)$, and the minor arc BC subtended the angle from the center of the circle is $\angle BOC$.

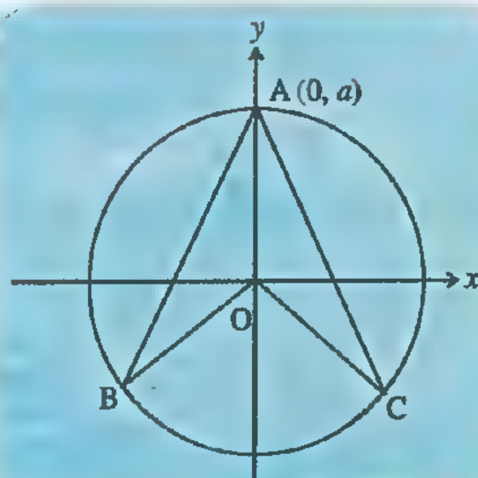


Fig. 7.16

If A(0,a) is a point on the major arc, then join AB and AC that develops the angle of the minor arc which is two times the angle subtended by the major arc:

$$\angle BOC = 2\angle BAC \qquad (61)$$

From the Fig. 7.16, if $\angle BAC = \theta$, then $\angle BOC = 2\theta$ and the result (61) can be verified as follows:

If the slopes of BA and AC are

$m_1 = \frac{a + y_1}{x_1}$, $m_2 = \frac{-y_1 - a}{x_1} = \frac{-(a + y_1)}{x_1}$, then, the angle $\angle BAC = \theta$ from BA to AC is:

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{\frac{a+y_1}{x_1}+\frac{a+y_1}{x_1}}{1-\frac{a+y_1}{x_1}\cdot\frac{a+y_1}{x_1}}=\frac{2(a+y_1)}{x_1}\cdot\frac{x_1^2}{x_1^2-(a+y_1)^2}$$

$$=\frac{2x_1(a+y_1)}{x_1^2-a^2-y_1^2-2ay_1}$$

$$=\frac{2x_1(a+y_1)}{-2y_1^2-2ay_1}=\frac{2x_1(a+y_1)}{-2y_1(a+y_1)}=-\frac{x_1}{y_1},\quad x_1^2+y_1^2=a^2 \qquad (62)$$

If the slopes of BO and CO are

$m_3=\frac{y_1}{x_1}$, $m_4=\frac{y_1}{-x_1}$, then, the angle $\angle BOC=2\theta$ from BO to OC is:

$$\tan 2\theta=\frac{m_3-m_4}{1+m_3m_4}=\frac{\frac{y_1}{x_1}+\frac{y_1}{x_1}}{1-\frac{y_1}{x_1}\cdot\frac{y_1}{x_1}}=\frac{2x_1^2y_1}{x_1(x_1^2-y_1^2)}=\frac{2x_1y_1}{x_1^2-y_1^2}$$

The trigonometric identity

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{\frac{-2x_1}{y_1}}{1-\frac{x_1^2}{y_1^2}}=\frac{-2x_1y_1}{y_1^2-x_1^2}=\frac{-2x_1y_1}{-(x_1^2-y_1^2)}=\frac{2x_1y_1}{x_1^2-y_1^2}$$

is proving result (63) with result (62). Thus $\angle BOC=2\angle BAC.$ (63)

Example 7.4.3:[Angle Subtended by Minor Arc]: Show that the angle subtended by the minor arc BC of the circle $x^2+y^2=9$ is two times the angle subtended in the major arc. The coordinates of the minor arc are $B(2,\sqrt{5}), C(2,-\sqrt{5})$.

Solution: The circle $x^2+y^2=9$, whose center is O(0,0). The arc BC is the minor arc of the given circle, whose coordinates are $B(2,\sqrt{5}), C(2,-\sqrt{5})$ and the minor arc BC subtended the angle from the center of the circle is $\angle BOC$.

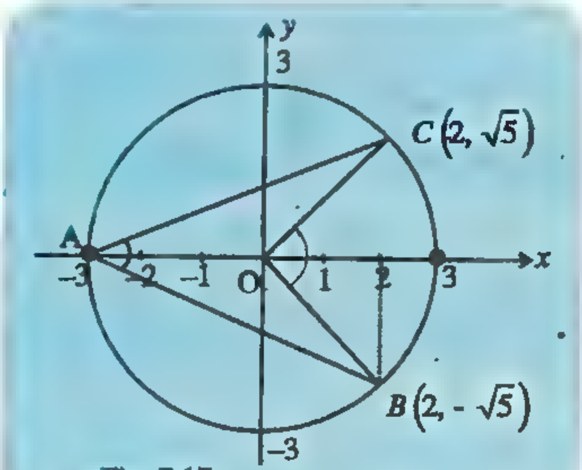


Fig. 7.17

If A(–3,0) is a point on the major arc, then join AB and AC that develops the angle of the minor arc which is two times the angle subtended by the major arc:

$$\angle BOC = 2\angle BAC \qquad (64)$$

From the Fig. 7.17, if $\angle BAC = \theta$ and $\angle BOC = 2\theta$, then, the result (64) can be verified as follows:

If the slopes of BA and AC are $m_1 = \frac{0+\sqrt{5}}{-3-2} = -\frac{1}{\sqrt{5}}$, $m_2 = \frac{\sqrt{5}-0}{2+3} = \frac{1}{\sqrt{5}}$,

then, the angle $\angle BAC = \theta$ from BA to AC is

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}}}{1 - \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}}} = \frac{-\frac{2}{\sqrt{5}}}{1 - \frac{1}{5}} = \frac{-\sqrt{5}}{2} \qquad (65)$$

If the slopes of BO and OC are

$m_3 = \frac{0+\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$, $m_4 = \frac{\sqrt{5}-0}{2} = \frac{\sqrt{5}}{2}$

then, the angle $\angle BOC = 2\theta$ from OC to BO is

$$\tan 2\theta = \frac{m_4 - m_3}{1 + m_3 m_4} = \frac{\frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2}}{1 - \frac{\sqrt{5}}{2} \cdot \frac{\sqrt{5}}{2}} = \frac{\sqrt{5}}{1 - \frac{5}{4}} = -4\sqrt{5}$$

The trigonometric identity

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2\left(\sqrt{5}/2\right)}{1 - \frac{5}{4}} = -4\sqrt{5} \qquad (66)$$

is proving result (65) with result (66). Thus $\angle BOC = 2\angle BAC$.

- ***An angle in a semicircle is a right angle***

Let the circle equation be

$$x^2 + y^2 = a^2 \qquad (67)$$

If $P(x_1, y_1)$ is any point on the semicircle and AB is fixed as the diameter of the circle (67) on the x-axis, whose coordinates are A $(a, 0)$ and B $(-a, 0)$, then the point $P(x_1, y_1)$ lies on the circle (67) that changes the circle equation to:

$$x_1^2 + y_1^2 = a^2$$

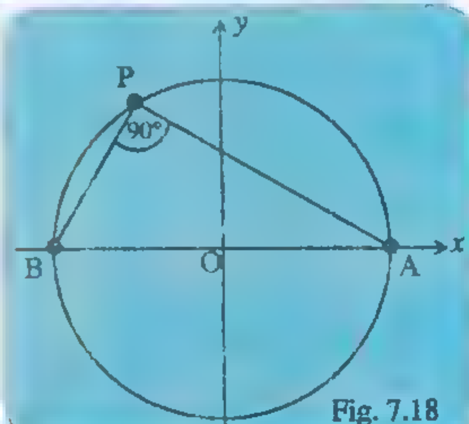


Fig. 7.18

Join PA and PB that develops a right angle $\angle APB$. The angle $\angle APB$ is a right angle, if AP and PB are perpendicular to each other, for which the slopes of AP and PB are respectively:

$$m_1 = \frac{y_1 - 0}{x_1 - a} = \frac{y_1}{x_1 - a}, \qquad m_2 = \frac{y_1 - 0}{x_1 + a} = \frac{y_1}{x_1 + a}$$

The product of the slopes of AP and BP is

$$m_1 m_2 = \left(\frac{y_1}{x_1 - a}\right)\left(\frac{y_1}{x_1 + a}\right) = \frac{y_1^2}{x_1^2 - a^2} = \frac{y_1^2}{-y_1^2} = -1, \quad x_1^2 + y_1^2 = a^2$$

Thus, PA and PB are perpendicular and the angle $\angle APB = 90°$ is of course a right-angle.

If $\angle APB = 90°$, then P is a point lies on the semicircle, for which the Pythagorean rule

$$|PA|^2 + |PB|^2 = |AB|^2 \qquad (68)$$

with substitution of

$$PA = (a - x_1, 0 - y)_1 \quad \Rightarrow |PA|^2 = \left[\sqrt{(a-x_1)^2 + y_1^2}\right]^2 = (a-x_1)^2 + y_1^2$$

$$PB = (-a - x_1, -0 - y_1) \quad \Rightarrow |PB|^2 = \left[\sqrt{(a+x_1)^2 + y_1^2}\right]^2 = (a+x_1)^2 + y_1^2$$

$$AB = (-a - a, 0 - 0) \quad \Rightarrow |AB|^2 = \left[\sqrt{(-2a)^2 + 0}\right]^2 = 4a^2$$

gives the locus of $P(x_1, y_1)$

$$|PA|^2 + |PB|^2 = |AB|^2,\ AB = (-a, 0) - (a, 0)$$

$$(a - x_1)^2 + y_1^2 + (a + x_1)^2 + y_1^2 = 4a^2$$

$$a^2 + x_1^2 - 2ax_1 + y_1^2 + a^2 + x_1^2 - 2ax_1 = 4a^2$$

$$2a^2 + 2x_1^2 + 2y_1^2 = 4a^2 \Rightarrow 2x_1^2 + 2y_1^2 = 2a^2 \Rightarrow x_1^2 + y_1^2 = a^2$$

which is a circle, P may lie on the upper or the lower semicircle.

Example 7.4.4: [Right Angle on a Semicircle]: Show that the angle in the semicircle of the circle $(x-h)^2 + y^2 = a^2$ is a right-angle.

Solution: The circle equation with center C (h, 0) is :

$$(x-h)^2 + y^2 = a^2$$

If $P(x_1, y_1)$ is any point on the semicircle and AB is fixed as the diameter of the given circle on the x-axis, then

$$(x_1 - h)^2 + y_1^2 = a^2$$

The coordinates of A and B are respectively:

$$OA = OC - AC = h - a \qquad \Rightarrow A(h-a, 0)$$

$$OB = OA + AB = (h-a) + 2a = h + a \quad \Rightarrow B(h+a, 0)$$

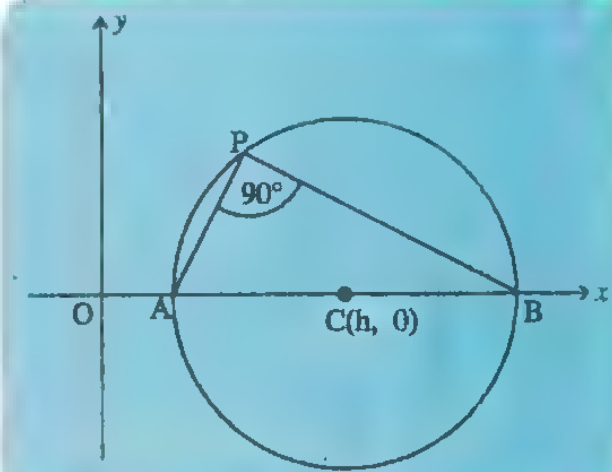


Fig. 7.19

Join PA and PB that develops a right angle $\angle APB$. This angle $\angle APB$ is a right angle, if AP and BP are perpendicular. They are perpendicular, if the product of their slopes equals -1:

$$m_1 m_2 = \frac{y_1}{x_1-h+a} \cdot \frac{y_1}{x_1-h-a} = \frac{y_1^2}{(x_1-h+a)(x_1-h-a)}$$

$$= \frac{y_1^2}{(x_1-h)^2-a^2} = \frac{y_1^2}{-y_1^2} = -1, \quad y_1^2 = a^2-(x_1-h)^2 = -\left[(x_1-h)^2-a^2\right]$$

$$\text{when } m_1 = \frac{y_1-0}{x_1-(h-0)} = \frac{y_1}{x_1-h+a} \quad \text{and } m_2 = \frac{y_1-0}{x_1-(h+a)} = \frac{y_1}{x_1-h-a}$$

Thus, the angle $\angle APB = 90°$ is right angle.

- ***The Perpendicular at the outer end of radial segment is tangent to the circle***

The circle equation with centre, $C(-g,-f)$ is:

$$x^2+y^2+2gx+2fy+c=0 \qquad (69)$$

If $P(x_1, y_1)$ is a point on the circle (69) and $C(-g,-f)$ is the centre of the circle (69), then CP is the radial segment of the circle (69).

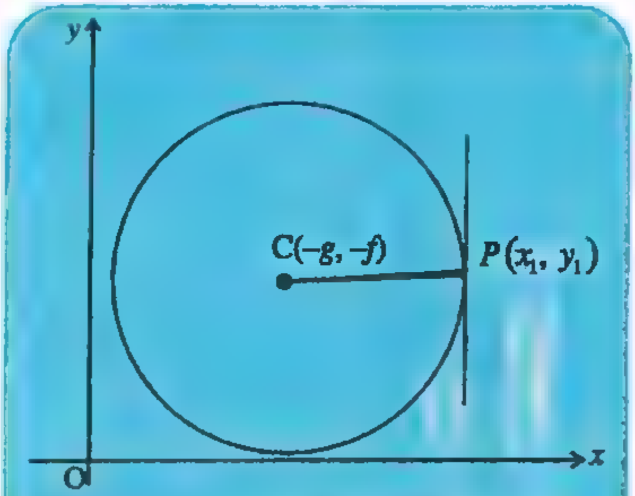


Fig. 7.20

The equation of the tangent line on the circle (69) at point P is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$
$$x(x_1 + g) + y(y_1 + f) + gx_1 + fy_1 + c = 0$$

whose slope is $m_1 = -\dfrac{x_1 + g}{y_1 + f}$ and the slope of CP is $m_2 = \dfrac{x_1 + f}{y_1 + g}$.

The perpendicular at the outer end P of the radial segment CP is tangent to the circle (69) if the product of the slopes of the radial segment CP and the line of the outer end of the radial segment CP is −1:

$$m_1 m_2 = -\frac{x_1 + g}{y_1 + f} \cdot \frac{y_1 + f}{x_1 + g} = -1$$

Thus, the perpendicular at the outer end P of the radial segment is tangent to the circle (69).

It is important to note that

- the tangent line is perpendicular to the radial segment, if the radial segment is the segment through the point of contact of the tangent and the center of the circle.
- if a line is perpendicular to the tangent of the circle at the point of contact, then it passes through the center of the circle:

Example 7.4.5: [Outer end Perpendicular]: Show that the perpendicular at the outer end point P(1,1) of the radial segment is tangent to the circle $x^2+y^2-13x-5y+16=0$.

Solution: The circle equation with center $C(13/2,5/2)$ is:

$$x^2+y^2-13x-5y+16=0$$

The equation of the tangent line on the given circle at a point $P(1,1)$ is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

$$x(1)+y(1)-\frac{13}{2}(x+1)+\frac{5}{2}(y+1)+16=0,\quad P(x_1,\ y)=P(1,1)$$

$$11x+3y-16=0$$

whose slope is $m_1=-11/3$ and the slope of the radial segment CP is

$$m_2=\frac{1-\frac{5}{2}}{1-\frac{13}{2}}=\frac{3}{11}$$

The product of the slopes of the tangent to the circle and the radial segment CP is

$$m_1m_2=\frac{-11}{3}\cdot\frac{3}{11}=-1$$

Thus, the perpendicular at the outer end P of the radial segment is tangent to the given circle at point P.

Exercise 7.3

1. If A(2,2) and B(3,1) are the end points of the chord AB of the circle $x^2+y^2-4x-2y+4=0$, then show that

   a. the line from the center of the circle is perpendicular to AB, also bisects the chord AB.

   b. the line from the center of the circle to the midpoint of the chord AB is perpendicular to the chord AB.

   c. the perpendicular bisector CD of the chord AB passes through the center of the given circle.

2. If A(0,0) and B(0,3) are the end points of the chord AB of the circle $x^2+y^2+4x-5y=0$, then show that

a. the line from the center of the circle is perpendicular to AB, also bisects the chord AB.

b. the line from the center of the circle to the midpoint of the chord AB is perpendicular to the chord AB.

c. the perpendicular bisector CD of the chord AB passes through the center of the given circle.

3. Show that the chords AB and DE are equidistant from the center C(0,0) of the circle

a. $x^2 + y^2 = 4$. The coordinates of the end points of the two chords AB and DE are A(–2,0) , B(0,2), D(0,2 and E(2,0).

b. $x^2 + y^2 = 16$. The coordinates of the end points of the two chords AB and DE are A(–4,0), B(0,4), D(0,4) and E(4,0).

c. $x^2 + y^2 =9$. The coordinates of the end points of the two chords AB and DE are A(–3,0), B(0,3), D(0, –3) and E(3,0).

4. Show that the angle subtended by the minor arc AB of the circle

a. $x^2 + y^2 = 9$ is two times the angle subtended in the major arc. The coordinates of the minor arc AB are A $(2, \sqrt{5})$, B $(2, -\sqrt{5})$.

b. $x^2 + y^2 = 4$ is two times the angle subtended in the major arc. The coordinates of the minor arc AB are A$(1, \sqrt{3})$, B $(1, -\sqrt{3})$.

c. $x^2 + y^2 = 16$ is two times the angle subtended in the major arc. The coordinates of the minor arc AB are A $(3, \sqrt{7})$, B $(-3, -\sqrt{7})$.

5. Show that the angle in the semicircle of the circle

a. $(x-h)^2 + y^2 = a^2,\ h=1,\ a=2$ is a right-angle.

b. $(x-h)^2 + y^2 = a^2,\ h=3,\ a=4$ is a right-angle.

c. $(x-h)^2 + y^2 = a^2,\ h=2,\ a=3$ is a right-angle.

Note that the diameter of the circle (in each case) is considered to be AB.

6. Show that the perpendicular at the outer end point

a. P(1,5) of the radial segment is tangent to the circle $x^2 + y^2 + x - 5y - 2 = 0$.

b. P(5,6) of the radial segment is tangent to the circle $x^2 + y^2 - 22x - 4y + 25 = 0$.

c. P (0, 0) of the radial segment is tangent to the circle $x^2 + y^2 - ax - by = 0$.

Glossary

- **Standard Form of a Circle:** The standard form of a circle with radius r and center C(h,k) is:

  $(x-h)^2+(y-k)^2=r^2$

- **General Form of a Circle:** The general form of a circle with radius $r=\sqrt{(-g)^2+(f)^2-c}$ and center C(-g,-f) is:

  $x^2+y^2+2gx+2fy+c=0$

  The coefficient of x^2 is equal to the coefficient of y^2, and there is no term containing xy and the square of the radius is $r^2 \geq 0$.

- **Nature of the circle:**

  If $g^2+f^2-c>0$, then the circle is real and different from zero.

  If $g^2+f^2-c=0$, then the circle shrinks to a point (-g,-f). It is called a point circle.

  If $g^2+f^2-c<0$, then the circle is imaginary or virtual.

- **Condition of Tangency:**

  The condition at which the line y=mx+c should touch the circle $x^2+y^2=a^2$ is:

  $c=\pm a\sqrt{1+m^2}$

  The equation of any tangent to the circle $x^2+y^2=a^2$ in the slope-form is:

  $y=mx\pm a\sqrt{1+m^2}$

  The condition at which the line lx+my+n=0 should touch the circle $x^2+y^2=a^2$ is:

  $n=\pm a\sqrt{l^2+m^2}$

  The condition at which the line lx+my+n=0 should touch the circle $x^2+y^2+2gx+2fy+c=0$ is:

  $(c-f^2)l^2+2fglm+(c-g^2)m^2-2n(gl+fm)+n^2=0$

- The tangent equation to the circle $x^2+y^2+2gx+2fy+c=0$ at a point $A(x_1, y_1)$ is:

  $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

  The tangent equation to the circle $x^2+y^2=a^2$ at a point $A(x_1, y_1)$ is:

  $xx_1+yy_1=a^2$

- The normal equation to the circle $x^2+y^2+2gx+2fy+c=0$ at a point $A(x_1, y_1)$ is:

  $x(y_1+f)-y(x_1+g)+(gy_1-fx_1)=0$

  The normal equation to the circle $x^2+y^2=a^2$ at a point $A(x_1, y_1)$ is:

  $xy_1-yx_1=0$

- The length of the tangent drawn from the point $P(x_1, y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$ is:

  $|PT|=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$

  The length of the tangent drawn from the point $P(x_1, y_1)$ to the circle $x^2+y^2=a^2$ is:

  $|PT|=\sqrt{x_1^2+y_1^2-a^2}$

  The lengths of the two tangents drawn from the point $P(x_1, y_1)$ on the given circle are equal.

UNIT

8

CONICS–II

This unit tells us, how to:

- define parabola and its elements focus, directrix, eccentricity, vertex focal chord (latus rectum).
- derive general form of an equation of parabola.
- derive the standard form of an equation of parabola and to sketch their graphical views.
- recognize the tangent and normal equations to parabola.
- find the condition at which a line is tangent to parabola and to determine the equation of a tangent line in slope–form.
- find the equations of tangent and normal to parabola.
- link parabola to our real–life situation problems.
- define ellipse and its elements vertices, foci, center, eccentricity, major and minor axes, focal chord (latus rectum).
- derive standard forms of ellipse and to sketch their graphical views.
- recognize the tangent and normal equations to ellipse.
- find the condition at which a line is tangent to ellipse and to determine the equation of a tangent line in slope–form.
- link parabola to our real–life situation problems.
- define hyperbola and its elements vertices, foci, center, eccentricity, transverse and conjugate axes, focal chord (latus rectum).
- derive standard forms of hyperbola and to sketch their graphical views.
- recognize the tangent and normal equations to hyperbola.
- find the condition at which a line is tangent to hyperbola and to determine the equation of a tangent line in slope–form.
- link hyperbola to our real–life situation problems.
- define the translation and rotation of axes.
- find the equations of transformation for translation and rotation of axes.
- find new origin and new axes referred to old origin and old axes.
- find the angle through which the axes be rotated about the origin so that the product term xy is removed from the translated equation.

8.1 Parabola

A topic in this unit involves the so–called conic sections: ellipses, parabolas and hyperbolas. All of these curves can be obtained by intersecting a plane with a right circular as discussed in the Unit–7.

The **conic** is that "the set of all points in the plane in such a way that its distances from a fixed point has a constant ratio to its distances from a fixed straight line."

For illustration, if F is a fixed point in the plane and L is the line in the same plane, then the set of all points P in the plane that satisfy the condition

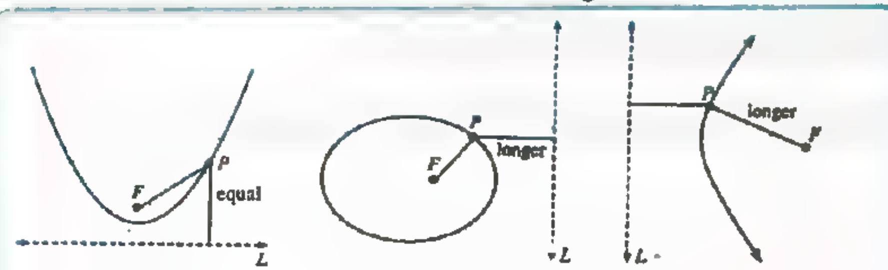
$$\frac{distance\ \ from\ P\ to\ F}{distance\ \ from\ P\ to\ L} = e \qquad (1)$$

is called a **conic section.** Here **e** is a fixed number for each conic called the **eccentricity** of the conic. The conic is an

ellipse, if $e < 1$
parbola, if $e = 1$
hyperbola, if $e > 1$

The fixed point is called the **focus**, the fixed straight line is called the **directrix** and the constant ratio is called the **eccentricity** of the conic and is denoted by e.

The criteria for a conic are illustrated in the Fig. 8.1:



Parabola if e=1 Ellipse if e < 1 Hyperbola if e > 1
Fig. 8.1

In this section, we look at one of these conic sections which is the parabola.

In our study of quadratic functions, the graph of the general form of the quadratic equation $y = ax^2 + bx + c$ (with $a \neq 0$) is a parabola that opens upward if a>0 and downward if a<0. Generally, the graph of a quadratic equation is always parabola, but not all parabolas can be represented by quadratic equation, because not all parabolas are graphs of functions.

Definition of parabola and its focus, directrix, eccentricity, vertex, focal chord and latus rectum

Definition 8.1.1:[Parabola]: The **parabola** is the set of all points P in the plane such that the distance from a fixed point F **(focus)** and the distance from a fixed straight line (**directrix**) to a point are **equidistant.**

The line through the focus perpendicular to the directrix is called the **principal axis** of the parabola, and the point where the axis intersects the parabola is called the

vertex. The line segment AB that passes through the focus perpendicular to the axis and with endpoints on the parabola is called the **focal chord** or its **latus rectum**. This terminology is shown in the figure (8.2).

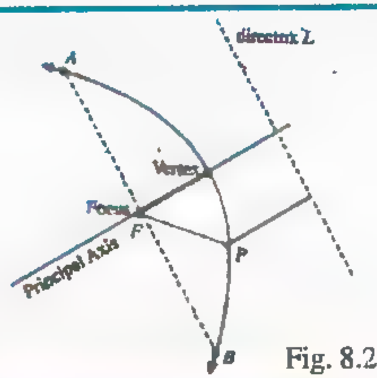


Fig. 8.2

8.7.1 General form of the parabola

i) Derivation of the general form of equation of parabola

To obtain the general form of the parabola, let us assume a focus with coordinates F(p,0) and a directrix $x=-p$, (p is any positive number) parallel to the y–axis. If P(x, y) is any point on the curve and $p_1(-p,y)$ is a point on the directrix $x=-p$, then by definition of parabola (figure 8.3)

$$\frac{\text{distance from } P \text{ to } F}{\text{distance from } P \text{ to } L}=e=1 \quad \text{for parabola } e=1$$

$$\text{distance from } P(x,y) \text{ to } L = \text{distance from } P(x,y) \text{ to } F$$

$$d(P,p_1)=d(P,F)$$

$$\sqrt{(x+p)^2+0}=\sqrt{(x-p)^2+(y-0)^2}, \qquad (2)$$

$$(x+p)^2=(x-p)^2+y^2, \quad \text{squaring}$$

$$x^2+2px+p^2=x^2-2px+p^2+y^2$$

$$4px=y^2 \qquad (3)$$

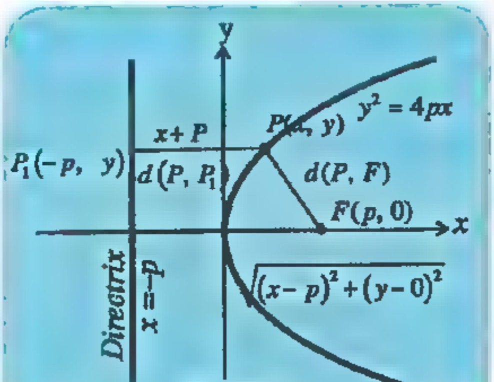


Fig. 8.3

The result (3) is the **standard form** obtained from the **general form** of the equation of a parabola with vertex at V(0,0), focus F(p,0) and directrix x= –p. The parabola is symmetric with respect to the positive x–axis if p>0 and symmetric with respect to the negative x–axis if p<0. The vertex V(0,0) of the parabola is on the principal axis of symmetry midway between the focus and the directrix.

8.1.2 Standard form of equation of parabola

Definition 8.1.2: [Parabola]: The standard form of the equation of a parabola that is symmetric with respect to the x–axis, with vertex V(0,0), focus F(p,0) and directrix the line x= –p is:

$$y^2 = 4px \qquad (4)$$

Definition 8.1.3:[Parabola]: The standard form of an equation of a parabola that is symmetric with respect to the y–axis, with vertex V(0,0), focus F(0,p) and directrix the line y= –p is:

$$x^2 = 4py \qquad (5)$$

The parabolas that have their vertex at the origin and open upward, downward, to the left and to the right are summarized in the table:

Parabola	Curve	Focus	Directrix	Vertex
$x^2 = 4py$	up, if p>0 down, if p<0	F(0,p) F(0,p)	y = -p y = -p	V(0,0) V(0,0)
$y^2 = 4px$	right, if p>0 left, if p<0	F(p,0) F(p,0)	x = -p x = - p	V(0,0) V(0,0)

ii) *Graphing standard form of a parabola*

Usually, the calculus is used in graphing parabolas. In this section, we will find and plot the parabola by inspection and count out units from the vertex in the appropriate direction as determined by the form of the equation. Finally, it is shown in the problem set that the length of the focal chord (latus rectum) is 4p. This number could be used in determination of the width of the parabola. This approach is employed in the following examples.

Example 8.1.1: [Graphing $y^2 = 4px$]: Graph the parabola $y^2 - 8x = 0$ and indicate the vertex, focus, directrix and the focal chord.

Solution: Rewrite the given parabola in the standard form

$$y^2 = 8x \qquad (6)$$

and is compared with the standard form of the parabola (2) to obtain :

$8 = 4p \Rightarrow p = 2$

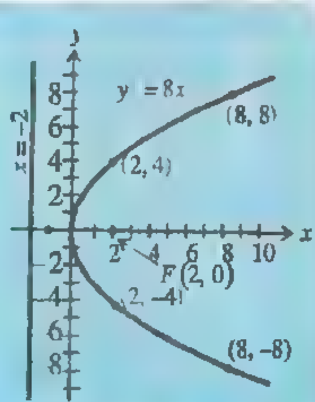


Fig. 8.4

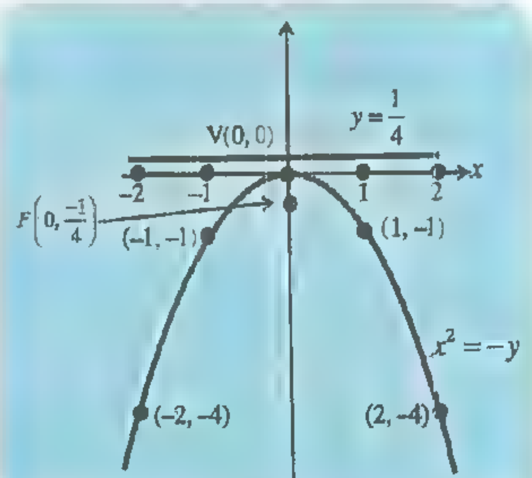


Fig. 8.5

Since p>0, the parabola opens to the right. The vertex is V(0,0), the focus is F(2,0), the directrix is the line x=-2 and the length of the focal chord is 4p=4(2)=8. The line of symmetry is the positive x–axis. This is shown in the figure (8.4).

Example 8.1.2: [Graphing $x^2 = 4py$]: Graph the parabola $x^2 + y = 0$ and indicate the vertex, focus, directrix and the focal chord.

Solution: Rewrite the given parabola in the standard form

$$x^2 = -y \qquad (7)$$

and is compared with the standard form of the parabola (5) to obtain :

$$-1 = 4p \Rightarrow p = -1/4$$

Since p<0, the parabola opens downward. The vertex is V(0,0), the focus is F(0,–1/4), the directrix is the line y=1/4 and the length of the focal chord is 4p=4(–1/4)= –1. The line of symmetry is the negative y–axis. This is shown in the figure (8.5).

iii) *The Equation of parabola with the given elements*

Example 8.1.3:[Equation of Parabola]: Find an equation of parabola with

a. focus F(0,–2) and directrix y=2.   b. vertex V(0,0) and focus F(5/8,0).

c. vertex V(0,0) and directrix x=1/2.

Solution:

a. By inspection, the value of p is p= –2 that satisfies the directrix y=2. This gives the equation of parabola $x^2 = 4py = 4(-2)y = -8y$, that opens downward (p<0) and the line of symmetry is the negative y–axis. This is shown in the figure (8.6):

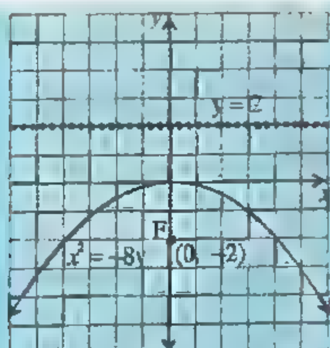


Fig. 8.6

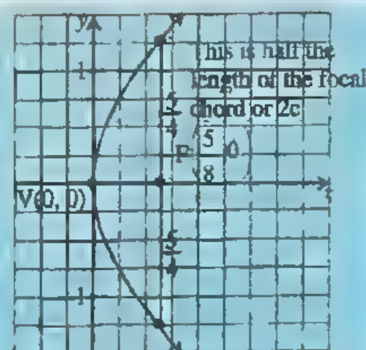


Fig. 8.7

b. By inspection, the value of p is p=5/8 that satisfies the directrix x=-5/8. This gives the equation of parabola $y^2 = 4px = 4(5/8)x = \frac{5}{2}x$, that opens right (p>0) and the line of symmetry is the positive x–axis. This is shown in the figure (8.7).

c. By inspection, the value of p is p= –1/2 that satisfies the directrix x=1/2. This gives the equation of parabola $y^2 = 4px = 4(-1/2)x = -2x$, that opens left

($p<0$) and the line of symmetry is the negative x–axis. This is shown in the figure (8.8):

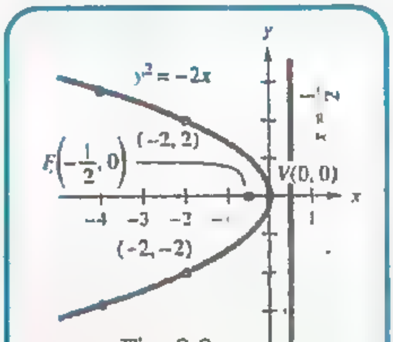


Fig. 8.8

Translation of Parabola: The replacement of X=x-h and Y=y-k in an equation of the standard form of parabola has the effect of translating the graph of an equation in the XY-plane,

- h units horizontally right if $h>0$ and left if $h<0$
- k units vertically up if $k>0$ and down if $k<0$

Important: Translation process is available in detail in sub-unit 8.4.

Definition 8.1.4 :[Translation of Parabola $y^2=4px$]: The standard form of the equation of a parabola that is symmetric with respect to the line y=k and with vertex V(h,k), focus F(h+p,k) and directrix line x=h−p is:

$$(y-k)^2=4p(x-h), \qquad (8)$$

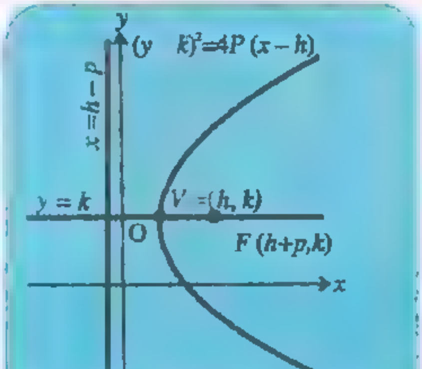


Fig. 8.9

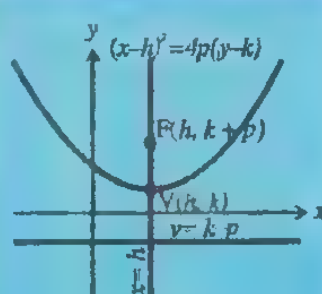


Fig. 8.10

Definition 8.1.5:[Translation of Parabola $x^2 = 4py$]: The standard form of the equation of a parabola that is symmetric with respect to the line x=h and with vertex V(h,k), focus F(h,k+p) and directrix line y=k–p is:

$$(x-h)^2 = 4p(y-k), \quad \text{Fig. 8.10} \qquad (9)$$

The parabolas in definitions (8.1.4) and (8.1.5) open to the right or up if p is positive. They open to the left or down if p is negative. The length of the focal chord is 4p and the end points of the focal chord are at a distance of 2p from the focus.

Example:8.1.4:[Graphing $(y-k)^2 = 4p(x-h)$]: Graph the parabola $(y+1)^2 = 6(x-2)$ and indicate the vertex, focus, directrix and the focal chord.

Solution: The given parabola

$$(y+1)^2 = 6(x-2) \qquad (10)$$

with substitution of

$$X=x-h=x-2 \text{ and } Y=y-k=y+1,\ h=2,\ k=-1 \qquad (11)$$

gives the new parabola in the XY–system:

$$Y^2 = 6X \qquad (12)$$

The parabola (12) is compared to the standard form parabola (4) to obtain:

$4p = 6 \Rightarrow p = 6/4 = 3/2$

Since p>0, the parabola opens to the right. The vertex of the new parabola (12) is V(0,0), the focus is F(3/2,0), the directrix is the line X=–3/2 and the length of the focal chord is 4p=4(3/2)=6. The line of symmetry is the positive X–axis. This is shown in Fig. 8.11.

The vertex, focus , directrix and the focal chord of the given parabola (10) are the following:

- The coordinates of the vertex V(0,0) of the new parabola are X=0 and Y=0. Put X=0 and Y=0 in (11) to obtain the coordinates of the vertex of the given parabola (10):

  X=x–2 => 0=x–2 => x=2 and Y=y+1 => 0=y+1 y=–1

  The vertex of the given parabola (10) is therefore V(2,–1). This vertex is agreed to the vertex of the standard parabola (4).

- The coordinates of the focus F(3/2,0) of the new parabola are X=3/2 and Y=0. Put X=3/2 and Y=0 in (11) to obtain the coordinates of the focus of the given parabola (10):

  X=x–2 => 3/2=x–2 => x=7/2 and Y=y+1 => 0=y+1 y=–1

The focus of the given parabola (10) is therefore F(7/2,–1)=F(h+p,k). This focus is agreed to the focus of the standard parabola (4).

- The directrix of the new parabola is X=–3/2. Put X= –3/2 in (11) to obtain the directrix of the given parabola (10):

  X=x–2 => –3/2=x–2 => x=1/2

  The directrix of the given parabola (10) is therefore x=1/2=h–p. This directrix is agreed to the directrix of the standard parabola (4).

- The length of the new focal chord and the given parabola (10) is of course 6. The focal chord is on the line through the focus F(7/2,–1) and the end points of the focal chord are 2p=2(3/2)=3 units from the focus F(7/2,–1). The coordinates of the focal chord are therefore:

$$\left(\frac{7}{2},-1+3\right)=\left(\frac{7}{2},2\right) \text{ and } \left(\frac{7}{2},-1-3\right)=\left(\frac{7}{2},-4\right)$$

- The line of symmetry is therefore y= –1.

  The graph of the parabola is shown in the figure (8.11).

The translated parabolas that have their vertex at V(h,k) and open upward, downward, to the left and to the right are summarized in the box:

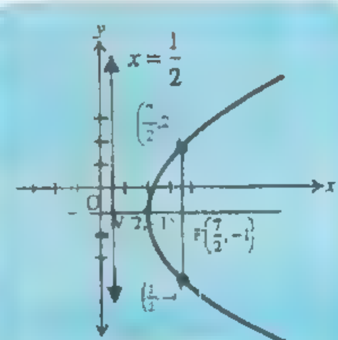


Fig. 8.11

Parabola	Curve	Focus/Vertex	Directrix
$(x-h)^2=4p(y-k)$	up, if p>0 down, if p<0	F(h,k+p), V(h,k)	y=k–p
$(y-k)^2=4p(x-h)$	right, if p>0 left, if p<0	F(h+p,k), V(h,k)	x=h–p

8.1.3 Equations of tangent and normal

i) Recognition of tangent and normal to parabola

A line which is parallel to the axis of a parabola intersects the parabola in only one (finite) point; all other lines will cut the parabola in two real and distinct points, real and coincident points, or complex conjugate points. A line which meets a parabola in two coincident points is called a tangent. A tangent to any curve at a point P is the

limiting position of a secant line, cutting the curve in two points P and Q as $Q \to P$. The normal can easily be shown in the subsection of this section.

ii) *The condition at which a line is tangent to parabola at a point*

The line is tangent to parabola, when the line intersects the parabola in **two real and coincident** points. The given parabola and line

$$y^2 = 4px \qquad (13)$$

$$y = mx + c \qquad (14)$$

develops a system of two equations:

$$\begin{aligned} y^2 &= 4px \\ y &= mx + c \end{aligned} \qquad (15)$$

The solution set $\{x, y\}$ of a system of equations (15) exists only, if the curves of the system (15) are intersecting. That set of points of intersection $\{x, y\}$ (a solution set) can be found by solving the nonlinear system (15) simultaneously.

The line (14) is used in parabola (13) to obtain the quadratic equation in x:

$$(mx + c)^2 = 4px$$

$$m^2x^2 + 2mcx + c^2 = 4px$$

$$m^2x^2 + 2x(mc - 2p) + c^2 = 0 \qquad (16)$$

The equation (16) being a quadratic equation in x, gives a set of two values x_1 and x_2 of x, which will be used in a line (14) to obtain a set of two y values y_1 and y_2.

Thus, a solution set $\{(x_1, y_1), (x_2, y_2)\}$ of the system (15) is of course a set of points of intersection of the system (15).

The points of intersection of the system (15) are real, coincident or imaginary, according as the roots of the quadratic equation (16) are real, coincident or imaginary or according as the discriminant of the quadratic equation (16):

$$Disc = 4(mc - 2p)^2 - 4m^2c^2 > 0, \quad \textit{real and different}$$

$$Disc = 4(mc - 2p)^2 - 4m^2c^2 = 0, \quad \textit{real and coincident}$$

$$Disc = 4(mc - 2p)^2 - 4m^2c^2 < 0, \quad \textit{imaginary}$$

Example 8.1.5:[Real and Coincident Points]: Under what condition the tangent line 4x-y-4=0 intersects the parabola $x^2 = y$?

Solution: The equations of the line and parabola are:

$$4x-y-4=0$$
$$y=4x-4 \qquad (17)$$
$$x^2=y \qquad (18)$$

The line (17) is used in parabola (18) to obtain the y–coordinates of the points of intersection:

$$x^2=y$$
$$x^2=4x-4, \quad y=4x-4$$
$$x^2-4x+4=0 \Rightarrow (x-2)^2=0 \Rightarrow x=2,2$$

The x–coordinates are used in the line (17) to obtain the y–coordinates y=4, 4

Thus, the set of two points of intersection (2,4) and (2,4) are real and distance and the tangent line 4x–y-4=0 is of course intersecting the parabola (18) at two coincident points (2,4) and (2,4).

iii) *The Equation of a tangent line in slope-form*

If m is the slope of the tangent to parabola

$$y^2=4px \qquad (19)$$

then the equation of that tangent line is of the form

$$y=mx+c \qquad (20)$$

Here c is to be calculated from the fact that the line (20) is tangent to parabola (19). The line (20) is used in parabola (19) to obtain the quadratic equation in x:

$$y^2=4px$$
$$(mx+c)^2=4px$$
$$m^2x^2+c^2+2mcx=4px$$
$$m^2x^2+2(mc-2p)x+c^2=0 \qquad (21)$$

If the line (20) touches the parabola (19), then the quadratic equation (21) has coincident roots for which the discriminant of the quadratic equation (21) equals zero:

$$4(mc-2p)^2-4(m^2)(c^2)=0$$
$$4m^2c^2+16p^2-16mcp-4m^2c^2=0$$
$$16p^2-16mcp=0$$
$$16p(p-mc)=0 \Rightarrow p-mc=0 \Rightarrow c=p/m \qquad (22)$$

The equation (22) represents the **condition of tangency**. The value of c from equation (22) is used in the line (20) to obtain the required equation of tangent:

$$y = mx + \frac{p}{m} \qquad (23)$$

It is important to note that

- the equation of any tangent to parabola $y^2 = 4px$ in the slope–form is:

  $$y = mx + \frac{p}{m}$$

- the line y=mx+c should touch the parabola $y^2 = 4px$ under condition:

$$y = mx + c = mx + \frac{p}{m},\ c = p/m,\ y^2 = 4px \qquad (24)$$

- the **condition of tangency** in case of parabola $x^2 = 4py$ and line y=mx+c is:

$$y = mx + c = mx - pm^2,\ c = -pm^2,\ x^2 = 4py \qquad (25)$$

Example 8.1.6:[Tangency Condition]: For what value of c, the line x–y+c=0 will touch the parabola $x^2 = 8y$? Use that value of c to find the tangent line that should touch the given parabola.

Solution: The value of c at which the line x–y+c=0 will touch the given parabola through result (25) is:

$$c = -pm^2 = -2(1) = -2,\ x^2 = 8y = 4(2)y,\ p = 2, m = 1$$

Here m is the slope of the line x–y+c=0. The required tangent line that should touch the parabola through (25) is:

$$y = mx + c$$
$$= x - 2 \Rightarrow x - y - 2 = 0$$

iv) *The Equation of a tangent line to parabola at a point*

Let the equation of the tangent line at a point $p(x_1, y_1)$ to parabola $y^2 = 4px$ be:

$$y - y_1 = m_1(x - x_1) \qquad (26)$$

Here m_1 is the slope of the tangent line to parabola $y^2 = 4px$ at a point $p(x_1, y_1)$ that can be found by differentiating $y^2 = 4px$ with respect to x:

$$y^2 = 4px$$

$$2y\frac{dy}{dx} = 4p \Rightarrow \frac{dy}{dx} = \frac{2p}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{2p}{y_1} = m_1, \text{ say} \quad (27)$$

The substitution of (27) in (26) is giving the equation of the tangent line at a point $p(x_1, y_1)$ to parabola $y^2 = 4px$:

$$y - y_1 = m_1(x - x_1)$$

$$y - y_1 = \frac{2p}{y_1}(x - x_1)$$

$$yy_1 - y_1^2 = 2px - 2px_1$$

$$yy_1 - 4px_1 = 2px - 2px_1, \quad y_1^2 = 4px_1$$

$$yy_1 = 2px + 2px_1$$

$$yy_1 = 2p(x + x_1) \quad (28\text{-a})$$

It is important to note that

- the equation of the tangent line at a point $p(x_1, y_1)$ to parabola $x^2 = 4py$ is:

$$xx_1 = 2p(y + y_1) \quad (28\text{-b})$$

- If the tangent line $y = mx + \frac{p}{m}$ to parabola $y^2 = 4px$ is identical to $yy_1 = 2p(x + x_1)$, then the coefficients of like terms of $y = mx + \frac{p}{m}$ and $yy_1 = 2p(x + x_1)$ are compared to obtain the contact point:

$$mx = \frac{2px}{y_1} \Rightarrow y_1 = \frac{2p}{m}$$

$$\frac{2px_1}{y_1} = \frac{p}{m} \Rightarrow 2x_1 = \frac{y_1}{m} \Rightarrow 2x_1 = \frac{2p}{m^2} \Rightarrow x_1 = \frac{p}{m^2}$$

Thus, the contact point is

$p(x_1, y_1) = \left(p/m^2, 2p/m\right)$ in case of parabola $y^2 = 4px$. (29)

- If the tangent line $y = mx - pm^2$ to parabola $x^2 = 4py$ is identical to $xx_1 = 2p(y + y_1)$, then the coefficients of like terms of $y = mx - pm^2$ and $xx_1 = 2p(y + y_1)$ are compared to obtain the point of contact:

$$mx = \frac{xx_1}{2p} \Rightarrow x_1 = 2pm$$

$$-pm^2 = -y_1 \Rightarrow y_1 = pm^2$$

Thus, the contact point is

$$p(x_1, y_1) = \left(2pm, pm^2\right) \text{ in case of parabola } x^2 = 4py. \qquad (30)$$

Example 8.1.7:[Tangent Equation]: Find the equation of tangent line at a point p(2,-4) to parabola $y^2 = 8x$. Show that p(2,-4) is the point of contact in between the required tangent line and the given parabola.

Solution: Result (28-a) is used to obtain the tangent line to the given parabola:

$$yy_1 = 2p(x + x_1)$$

$$y(-4) = 2(2)(x+2),\ p(x_1, y_1) = (2,-4),\ 4p = 8$$

$$-4y = 4x + 8$$

$$4x + 4y + 8 = 0$$

$$x + y + 2 = 0$$

The point of contact through result (29) is:

$p(x_1, y_1) = \left(p/m^2, 2p/m\right) = (2,-4),\ p = 2,\ m = -1$ is the slope of the tangent line x+y+2=0

v) *The Equation of a normal line to parabola at a point*

The equation of the normal line at a point $p(x_1, y_1)$ to parabola $y^2 = 4px$ is:

$$y - y_1 = m_2(x - x_1) \qquad (31)$$

Here m_2 is the slope of the normal line to parabola $y^2 = 4px$ at a point $p(x_1, y_1)$ that can be found by differentiating $y^2 = 4px$ with respect to x:

$$y^2 = 4px$$

$$2y\frac{dy}{dx} = 4p$$

$$\frac{dy}{dx} = \frac{2p}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{2p}{y_1} = m_1 \Rightarrow m_2 = \frac{-1}{m_1} = \frac{-y_1}{2p}, \text{ say} \qquad (32)$$

The substitution of (32) in (31) is giving the normal equation at a point $p(x_1, y_1)$ to parabola $y^2 = 4px$:

$$y - y_1 = m_2(x - x_1)$$

$$y - y_1 = \frac{-y_1}{2p}(x - x_1),\ m_2 = -y_1/2p \qquad (33)$$

Example 8.1.8:[Normal Equation]: Find the normal equation at a point p(2,–4) to parabola $y^2 = 8x$.

Solution: Result (33) is used to obtain the normal line to the given parabola:

$$y - y_1 = \frac{-y_1}{2p}(x - x_1)$$

$$y - (-4) = \frac{4}{2(2)}(x - 2),\quad P(x_1, y_1) = (2, -4), 4p = 8$$

$$y + 4 = x - 2$$

$$x - 2 - y - 4 = 0$$

$$x - y - 6 = 0$$

8.1.4 Application of parabola

The parabola is more than just a geometric concept. It has many uses in the physical world that are listed under:

1. Projectiles in the air, such as a ball, or a missile, or water sprayed from a hose, describe a parabolic path when acted on only by gravity.
2. Many arches of bridges or buildings are parabolic in shape. With this shape, the arch can support the structure above it.

Fig. 8.12

3. Rotating a parabola about its line of symmetry, creates a bowl type surface called a paraboloid of revolution. A paraboloid has an important reflection property. Any ray or wave that originates at the focus and strikes the surface of the paraboloid is reflected parallel to the line of symmetry. (See figure (8.12)).

This forms the basic design of the reflectors for automobile headlights, flashlights, searchlights, telescopes, etc. This is also an excellent collecting device and is the basic design of TV, radar, and radio antennas.

Suspension and reflection problems related to parabola

Example 8.1.9:[Cables on a Bridge]: The cables of a bridge form a parabolic arc. The low point of the cable is 10ft above the roadway midway between two towers. The distance between the towers is 400 ft. The cable is attached to the towers 50ft above the roadway. Determine the equation of the parabola that describes the path of the cable. This is shown in the figure (8.13) below:

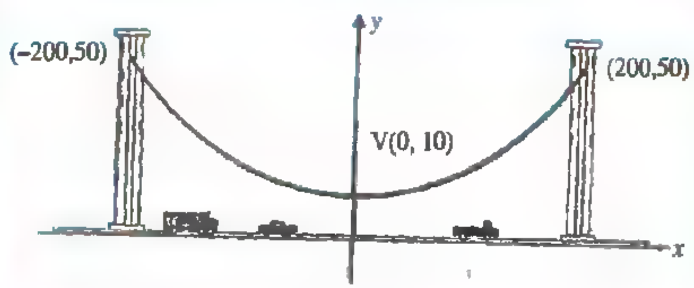


Fig. 8.13

Solution: The parabola is formed by the cable between the two towers. The low point on the cable is midway between the towers, and 10ft above the roadway. In order to write an equation, locate the x–axis and the y–axis in the xy–plane. Select the roadway as the x–axis and the line perpendicular to the roadway through the lowest point of the tower as the y–axis. The parabola opens up with the vertex at the point (0,10). Two other points on the parabola are (200,50) and (–200,50). The standard form for this equation is:

$$(x-h)^2 = 4p(y-k) \qquad (34)$$

The vertex V(0,10) and a point on the curve (x, y)=(200,50) are used in (34) to obtain p:

$$(x-h)^2 = 4p(y-k), \text{ translate } h \text{ units on the } x-\text{axis}, k \text{ units on the } y-\text{axis}$$

$$(200-0)^2 = 4p(50-10)\,), \quad V(h,k) = V(0,10), (x,y) = (200,50)$$

$$40000 = 160p \Rightarrow p = 250$$

The substitution of V(h,k)=(0,10) and p=250 in equation (34) is giving the parabolic equation

$$(x-0)^2 = 4(250)(y-10)$$

$$x^2 = 1000(y-10) \Rightarrow x^2 - 1000y + 10000 = 0$$

that describes the path of the cable.

Example 8.1.10:[Radar Antenna]: A radar antenna is constructed so that a cross section along its axis is a parabola with the receiver at the focus. Find the focus if the antenna is 12 m across and its depth is 4 m. Find the equation of parabola that described the radar antenna. This is shown in the Fig. 8.14:

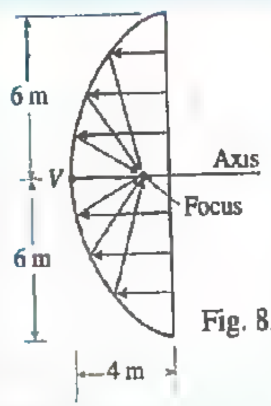


Fig. 8.14

Solution: The parabola is formed by the radar antenna. In order to write an equation, locate the x–axis and the y–axis in the xy–plane. The axis of symmetry is the positive x–axis. The parabola opens to the right with the vertex at the origin V(0,0). The other point on the parabola is (4, 6). The standard form for this equation is:

$$y^2 = 4px$$

$$36 = 4p(4), \quad (x,y) = (4,6) \tag{35}$$

$$p = \frac{36}{16} = \frac{9}{4}$$

Thus, the parabolic equation that describes the radar antenna is obtained by putting p=9/4 in (35):

$$y^2 = 4px = 4\left(\frac{9}{4}\right)x = 9x$$

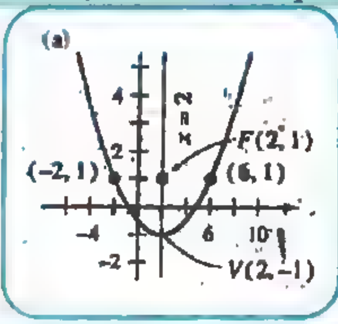
The focus is F(0,9/4) which is 9/4 m from the vertex V(0,0).

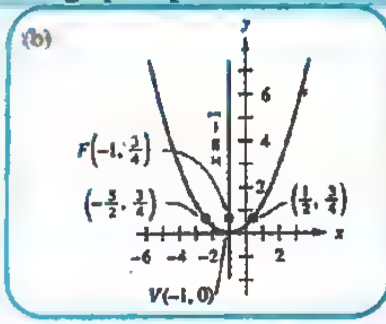
Exercise 8.1

1. In each case, sketch the parabola represented by the equation, indicate the vertex, the focus, the end points of the focal chord (latus rectum) and the axis of symmetry:

   a. $x^2 = 2y$ b. $y^2 = -3(x+1)$ c. $(y-3)^2 = x$

2. In each case, determine the equation of graphed parabola:





3. In each case, write the equation of parabola through the given information:

   a. Focus at F(0,3), directrix y= –3.

   b. Focus at F(4,0), directrix x= –4.

   c. Vertex at V(0,0), x–axis is the line of symmetry, passes through (3,6).

   d. Vertex at V(0,0), y–axis is the line of symmetry, passes through (–12,–3).

   e. Line of symmetry is vertical, passes through (–3,4), vertex at V(5,1).

   f. Line of symmetry is horizontal, passes through (7,9), vertex at V(3,–7).

4. Find the equation of the set of all points with distances from (4,3) that equal their distances from (–2,1).

5. Find an equation for a parabola whose focal chord has length 6, if it is known that the parabola has focus (4,–2) and its directrix is parallel to the y–axis.

6. In each case, find the points of intersection in between the line and the parabola:

   a. $y^2 = 9x,\ \ x - y + 2 = 0$ b. $y^2 + 3x = -8,\ \ x - y + 2 = 0$

c. $x^2 = 2y, \quad x - y - 2 = 0$ d. $y^2 = -2(x+1), \quad x + y - 2 = 0$

7. For what value of c,

a. the line x–y+c=0 will touch the parabola $y^2 = 9x$?

b. the line x–y+c=0 will touch the parabola $y^2 = 8x$?

c. the line x+y+c=0 will touch the parabola $x^2 = -8y$?

d. the line x–y+c=0 will touch the parabola $x^2 = \frac{2}{3}y$?

8. In each case, find the tangent equation and normal equation

a. at a point (1,2) to parabola $y^2 = 4x$.

b. at a point (3,6) to parabola $y^2 = 12x$.

c. at a point (–3/4,3) to parabola $y^2 = -12x$.

d. at a point (1/2,–1/3) to parabola $x^2 = -3y/4$.

9. Find the tangent equation

a. to parabola $y^2 = 8x$ which is perpendicular to the line 2y–x+1=0. Find also the point of contact.

b. to parabola $y^2 = 5x$ which is parallel to the line x+4y+1=0. Find also the point of contact.

10. Find the tangent equation

a. to parabola $y^2 = x$, which makes an angle of 45^0 with the x–axis.

b. to parabola $y^2 = x$ which makes an angle of 60^0 with the x–axis.

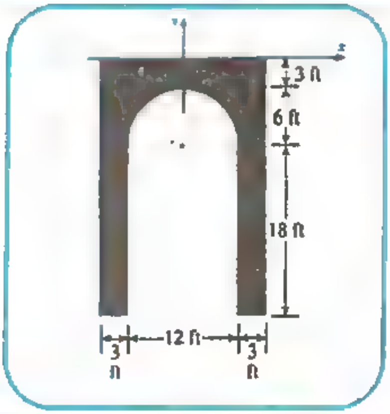
c. to parabola $y^2 = x$, which makes an angle of 135^0 with the x–axis.

11. Find the tangent equation

a. to parabola $x^2 = y$, which makes an angle of 45^0 with the x–axis.

b. to parabola $x^2 = y$ which makes an angle of 60^0 with the x–axis.

12. Find the equation of the parabolic portion of the archway, if parabolic archway has the dimensions shown in the figure below:



8.2 Ellipse

In shape and in format, the ellipse is different from the parabola. Although the parabola is open at one end, the ellipse is entirely closed. The parabola has one focus and one vertex, while the ellipse has two foci (plural of focus) and two vertices.

i) Definition of ellipse and its center, foci, vertices, eccentricity, focal chord, major and minor axes

Definition 8.2.1:[Ellipse]: An **ellipse** is the set of all points in the plane such that the sum of the distances from two fixed points (foci) to a particular point on the ellipse hold off a constant quantity.

The two fixed points $F_1(-c,0)$ and $F_2(c,0)$ are called **the foci** that lie on the x–axis at a distance **c** on each side of the origin. The ellipse crosses the x–axis at two points $V_1(-a,0)$ and $V_2(a,0)$ that are at a distance **a** on each side of the origin. The ellipse also crosses the y–axis at two points $B_1(0,b)$ and $B_2(0,-b)$ that lie at a distance **b** above and below the origin. This is shown in the figure (8.15).

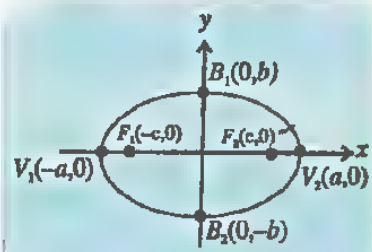


Fig. 8.15

The portion of the x–axis bounded by the ellipse is its **major axis**, and the distance **a** is called the length of the **semimajor axis**. The end points of the major axis

are called the **vertices** of the ellipse. The portion of the y–axis bounded by the ellipse is its **minor axis**, and the distance **b** is called the length of the **semiminor axis**. The center is the midpoint of the line segment F_1F_2. The ellipse is symmetrical with respect to the major and minor axes.

By definition, the sum of the distances from the two foci $F_1(-c,0)$, $F_2(c,0)$ to the point $V_2(a,0)$ is:

$$d(F_1,V_2)+d(F_2,V_2)=(a+c,0)+(a-c,0)$$
$$=(a+c+a-c,0)=(2a,0)$$

whose length is

$$|d(F_1,V_2)+d(F_2,V_2)|=\sqrt{(2a)^2+0}=\left[(2a)^2\right]^{1/2}=2a \qquad (35)$$

Similarly, the sum of the distances from the two foci $F_1(-c,0)$, $F_2(c,0)$ to the point $B_1(0,b)$ is:

$$d(F_1,B_1)+d(F_2,B_1)=(0+c,b-0)+(0-c,b-0)$$
$$=(c,b)+(-c,b)$$

$$|d(F_1,B_1)|+|d(F_2,B_1)|=\sqrt{c^2+b^2}+\sqrt{c^2+b^2}$$
$$=2\sqrt{c^2+b^2}=2a \qquad (36)$$

Result (36) is verified by substitution:

$$2\sqrt{c^2+b^2}=2a \Rightarrow 4(c^2+b^2)=4a^2 \Rightarrow c^2+b^2=a^2 \Rightarrow c^2=a^2-b^2$$

ii) General form of the ellipse

If P(x, y) is any point on the ellipse, then the distances from the two foci $F_1(-c,0)$ and $F_2(c,0)$ to the point P(x,y) are the following:

$$d(F_1,P)=(x+c,y-0)$$
$$|d(F_1,P)|=\sqrt{(x+c)^2+(y)^2}$$
$$d(F_2,P)=(x-c,y-0)$$
$$|d(F_2,P)|=\sqrt{(x-c)^2+(y)^2}$$

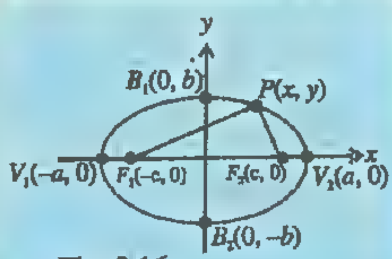


Fig. 8.16

By definition of an ellipse , the general form of an ellipse is:

$$|d(F_1,P)|+|d(F_2,P)|=2a$$

$$\sqrt{(x+c)^2+(y)^2}+\sqrt{(x-c)^2+y^2}=2a$$

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

Squaring both sides to obtain

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$4a\sqrt{(x-c)^2+y^2}=4a^2-4cx,$$

$$a\sqrt{(x-c)^2+y^2}=a^2-cx$$

Again squaring to obtain

$$a^2[(x-c)^2+y^2]=a^4-2a^2cx+c^2x^2$$

$$a^2\left(x^2-2cx+c^2+y^2\right)=a^4-2a^2cx+c^2x^2$$

$$a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2$$

$$a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$$

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$$

$$b^2x^2+a^2y^2=a^2b^2, \qquad a^2-c^2=b^2,\ a>0, b>0$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \textit{divide out by } a^2b^2 \qquad (37)$$

iii) *Circle is a special case of an ellipse*

The relative shape of an ellipse can be determined by its eccentricity e. The distance from the center of the ellipse to a focs is c, and the distance from the center to a vertex is a. The eccentricity is given by the equation:

$$e=\frac{c}{a} \qquad (38)$$

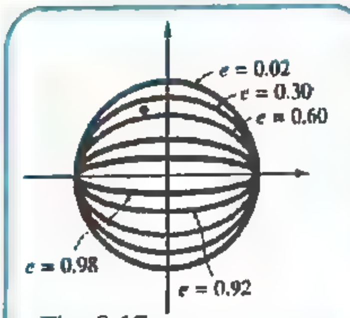


Fig. 8.17

The eccentricity of all ellipses are in a range between 0 and 1 (0<e<1). This is shown in the fig 8.17.

An ellipse with an eccentricity close to 1 is long and thin, and the foci are relatively far apart. If the eccentricity is small, close to 0, then the ellipse

resembles a circle. It can be shown that the circle is a special case of the ellipse when e=0.

8.2.1 Standard form of equation of an ellipse

Definition 8.2.2:[Standard Form of Horizontal Ellipse]: The standard form of the equation of an ellipse with center at the origin, length of the semimajor axis a, length of the semiminor axis b and major axis along the x–axis is shown in the figure (8.18):

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (39)$$

If we replace the foci on the y–axis, center at the origin, and pick any point P(x, y) on the ellipse, then we can develop the equation of the vertical ellipse given in definition 8.2.3.

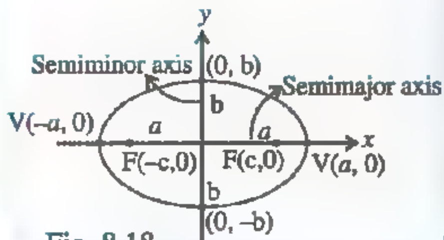


Fig. 8.18

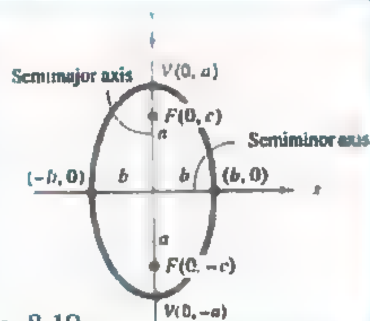


Fig. 8.19

Definition 8.2.3:[Standard Form of vertical Ellipse]: The standard form of the equation of an ellipse with center at the origin, length of the semimajor axis a, length of the semiminor axis b and major axis along the y–axis is shown in the Fig. 8.19:

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \qquad (40)$$

Graphing Ellipse: For graphing an ellipse, it is required to plot the center, the intercepts $\pm a$ on the major axis and $\pm b$ on the minor axis.

First, rewrite the equation of the ellipse in the standard form, so that there is a "1" on the right and the numerator coefficients of the square terms are also 1. The center is at (0,0) and plot the intercepts on the x–axis and y–axis. For the x–intercepts,

plot $\pm$ the square root of the number a^2; for the y–intercepts, plot $\pm$ the square root of the number b^2; finally, and draw the ellipse using these intercepts. The longer axis is called the **major axis**. If this larger axis is horizontal, then the ellipse is called **horizontal**, and if the major axis is vertical, the ellipse is then called **vertical**.

The orientation of the ellipse equation with center C(0,0), vertices/end points of the major axis and the end points of the semiminor axis are summarized in the box:

Orientation	Foci	Vertices/Semimajor [redacted]	Semiminor axis
Horizontal	(–c,0), (c,0)	(–a,0), (a,0)	(0,b), (0,–b)
Vertical:	(0,c), (0,–c)	(0,-a), (0,a)	(b,0), (–b,0)
Here $a^2=b^2+c^2$ or	$c^2=a^2-b^2$	a>b>0	

Example 8.2.1:[Vertical Ellipse]: Determine the vertices, end points of the minor axis and the coordinates of foci of the ellipse $9x^2+4y^2=36$. Sketch the developed ellipse.

Solution: Rewrite the ellipse equation in the standard form:

$$9x^2+4y^2=36$$

$$\frac{9x^2}{36}+\frac{4y^2}{36}=1, \quad \textit{divide out by 36}$$

$$\frac{x^2}{4}+\frac{y^2}{9}=1 \qquad (41)$$

The equation (41) is related to the vertical standard form ellipse (40). The center of the ellipse is at the origin, but the vertices of the major axis are on the y–axis, since the larger numerical value is under y^2. Thus, $a^2=9$ or a=3, and $b^2=4$ or b=2 and $c^2=a^2-b^2=9-4=5$ or $c=\pm\sqrt{5}$.

The coordinates of the center, vertices/end points of the major axis, end points of the minor axis and the foci are the following:

$C(0,0)$	center
$V_1(0,3), V_2(0,-3)$	end points of the major axis
$B_1(2,0), B_2(-2,0)$	end points of the minor axis

$F_1(0,\sqrt{5}),\ F_2(0,-\sqrt{5})$ foci

For some points on the ellipse,

when y=1, then $\frac{x^2}{4}+\frac{1}{9}=1 \Rightarrow x=\pm 4\frac{\sqrt{2}}{3}$

when y=2, then $\frac{x^2}{4}+\frac{4}{9}=1 \Rightarrow x=\pm 2\frac{\sqrt{5}}{3}$

The ellipse is symmetrical with respect to the major axis, minor axis. The center, vertices, foci, and the points

$$\left(\frac{4\sqrt{2}}{3},-1\right),\left(\frac{-4\sqrt{2}}{3},\right),\left(\frac{-4\sqrt{2}}{3},-1\right),\left(\frac{2\sqrt{5}}{3},-2\right)\left(\frac{-2\sqrt{5}}{3},2\right)\left(\frac{-2\sqrt{5}}{3},-2\right)$$

are labeled to obtain the graph of the given ellipse:

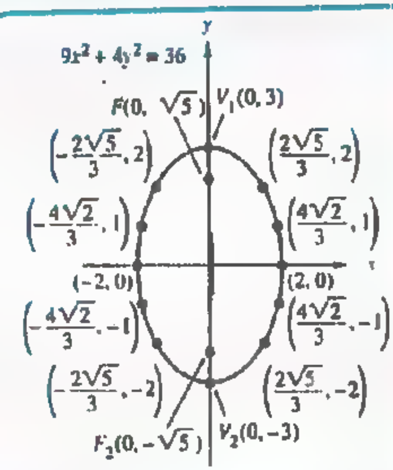


Fig 8.20

Example 8.2.2:[Horizontal Ellipse]: Determine the vertices, end points of the minor axis and the coordinates of foci of the ellipse $2x^2+5y^2=10$.

Solution: Rewrite the ellipse equation in the standard form:

$$2x^2+5y^2=10$$

$$\frac{2x^2}{10}+\frac{5y^2}{10}=1, \quad \textit{divide out by 10}$$

$$\frac{x^2}{5}+\frac{y^2}{2}=1$$

$$\frac{x^2}{(\sqrt{5})^2}+\frac{y^2}{(\sqrt{2})^2}=1 \qquad (42)$$

The ellipse (42) is related to the horizontal standard form ellipse (39). The center of the ellipse is at the origin, but the vertices of the major axis are on the x–axis, since the larger numerical value is under x^2. Thus, $a^2=5$ or $a=\sqrt{5}$ and $b^2=2$ or $b=\sqrt{2}$ and $c^2=a^2-b^2=5-2=3$ or $c=\pm\sqrt{3}$.

The coordinates of the center, vertices/end points of the major axis, end points of the minor axis and the foci are the following:

$C(0,0)$	center
$V_1(-\sqrt{5},0), V_2(\sqrt{5},0)$	end points of the major axis
$B_1(0,\sqrt{2}), B_2(0,-\sqrt{2})$	end points of the minor axis
$F_1(-\sqrt{3},0), F_2(\sqrt{3},0)$	foci

i) *Equation of an ellipse through its elements*

Example 8.2.3:[Equation of Ellipse]: Find an equation for the ellipse with foci $F_1(-1,0)$ and $F_2(1,0)$ and vertices $V_1(-2,0)$ and $V_2(2,0)$.

Solution: By inspection, the center of the ellipse is at C(0,0) and the distance from the center to the vertex is a=2; and the distance to a focus is c=1. The value of b is obtained by inserting a and c in the equation:

$$b^2=a^2-c^2=4-1=3 \Rightarrow b=\pm\sqrt{3}$$

The values of a and b are used in the horizontal standard form ellipse (39) to obtain

$$\frac{x^2}{4}+\frac{y^2}{3}=1 \qquad (43)$$

The procedure of translation of axes introduced in section 8.1 can also be utilized for ellipses and hyperbolas.

ii) Conversion of an equation to the standard form equation of ellipse

Definition 8.2.4:[Translation of Ellipse Horizontally]: The standard form of the equation of an ellipse with center at C(h,k), length of the semimajor axis a and semiminor axis b, and major axis parallel to the x–axis is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1, \quad \text{Fig. 8.21} \qquad (44)$$

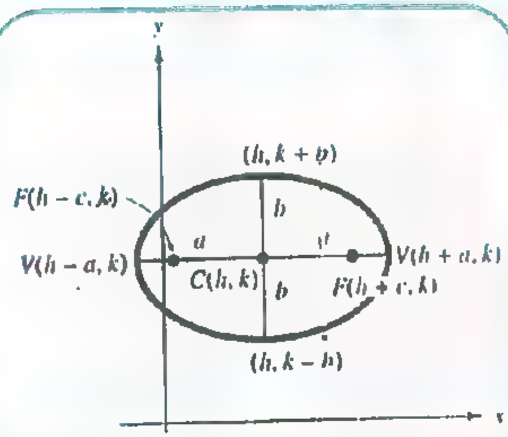


Fig 8.21

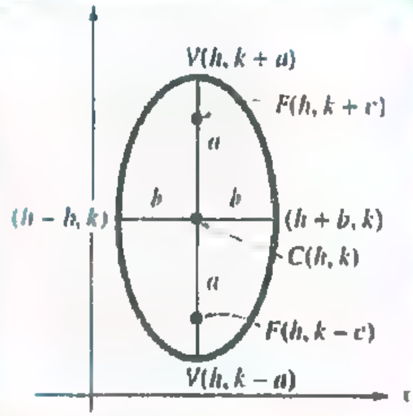


Fig. 8.22

Definition 8.2.5:[Translation of Ellipse Vertically]: The standard form of the equation of an ellipse with center at C(h,k), length of the semimajor axis a and semiminor axis b, and major axis parallel to the y–axis is:

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1, \quad \textit{Fig}.8.22 \qquad (45)$$

Example 8.2.4:[Translation of Ellipse]: Graph the ellipse whose equation is $4x^2+25y^2-8x+100y+4=0$. Indicate the center, vertices, foci and the end points of the minor axis.

Solution: Rewrite the given ellipse equation to the standard form by completing square:

$$4x^2-8x+25y^2+100y=-4$$

$$4(x^2-2x+1)+25(y^2+4y)=0$$

Add 100 with both sides to obtain

$$4\left(x^2-2x+1\right)+25\left(y^2+4y+4\right)=100$$

$$4(x-1)^2+25(y+2)^2=100$$

$$\frac{4(x-1)^2}{100}+\frac{25(y+2)^2}{100}=1$$

$$\frac{(x-1)^2}{25}+\frac{(y+2)^2}{4}=1 \qquad (46)$$

The given ellipse (46) with substitution X=x–h=x–1 and Y=y–k=y+2, h=1, k= –2 gives the translated ellipse in the XY-plane:

$$\frac{X^2}{25}+\frac{Y^2}{4}=1 \qquad (47)$$

The center of the ellipse is at the origin. The major axis is horizontal and the vertices are on the x–axis. Thus, a=5, b=2 and $c=\pm\sqrt{21}=\pm 4.58$.

The coordinates of the center, vertices/end points of the major axis, end points of the minor axis and the foci of the translated ellipse (47) are the following:

$C(0,0)$	center
$V_1(-5,0), V_2(5,0)$	end points of the major axis
$B_1(0,2), B_2(0,-2)$	end points of the minor axis
$F_1(-4.58,0), F_2(4.58,0), \pm\sqrt{21}\sqrt{21}=\pm 4.58$	foci

The coordinates of the center, vertices/end points of the major axis, the end points of the minor axis and foci of the given ellipse (46) are the following:

- The coordinates of the center $C(0,0)$ of the translated ellipse are X=0, Y=0. Put X=0 and Y=0 in (47) to obtain the coordinates of the center of the given ellipse (46):
  X=x–1 => 0=x–1 => x=1 and Y=y+2 => 0=y+2 => y= –2
  The center of the given ellipse (46) is C(1,–2).
- The coordinates of the vertices $V_1(-5,0), V_2(5,0)$ of the translated ellipse are X=–5, Y=0 (in case of V_1) and X=5, Y=0 (in case of V_2). Put X= –5 and Y=0 in (47) to obtain the coordinates of the vertex V_1 of the given ellipse (46):
  X=x–1 => –5=x–1 => x=–4 and Y=y+2 => 0=y+2 => y= –2
  The vertex V_1 of the given ellipse (46) is $V_1(-4,-2)$ and the vertex V_2 of the given ellipse (46) is of course $V_2(6,-2)$.

- The coordinates of the foci $F_1(-4.58,0)$, $F_2(4.58,0)$ of the translated ellipse are X= –4.58, Y=0 (in case of F_1) and X=4.58, Y=0 (in case of F_2). Put X= –4.58 and Y=0 in (47) to obtain the coordinates of the focus F_1 of the given ellipse (46):

X=x–1 => –4.58=x–1 => x=–3.58 and Y=y+2 => 0=y+2 => y=–2
The focus F_1 of the given ellipse (46) is $F_1(-3.58,-2)$ and the focus F_2 of the given ellipse (46) is of course $F_2(5.58,-2)$.

The graph of the ellipse is shown in the figure (8.23)

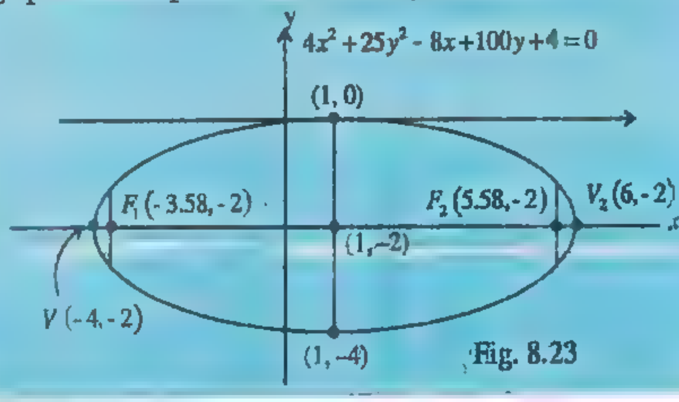


Fig. 8.23

The orientation of the ellipse equation with center C(h, k) is summarized in the boxes:

Orientation	Foci	Vertices/End Points of Major Axis	End Points of Minor Axis
Horizontal	$F_1(h-c,k), F_2(h+c,k)$	$V_1(h-a,k), V_2(h+a,k)$	$B_1(h,k+b), B_2(h,k-b)$
Vertical	$F_1(h,k+c), F_2(h,k-c)$	$V_1(h,k+a), V_2(h,k-a)$	$B_1(h+b,k), B_2(h-b,k)$

Note that $b^2=a^2-c^2$ or $c^2=a^2-b^2$ with a>b>0.

Example 8.2.5:[Equation of Ellipse]: Find the equation of the ellipse with vertices at (–1,2) and (7,2) and with 2 as the length of the semiminor axis.

Solution: With the vertices of the ellipse are at $V_1(-1,2)$ and $V_2(7,2)$, the center is at the midpoint of the line segment V_1V_2 joining these vertices. The midpoint (3,2) of the

line segment V_1V_2 is the center C(h,k)=C(3,2). This is shown in the figure (8.24).

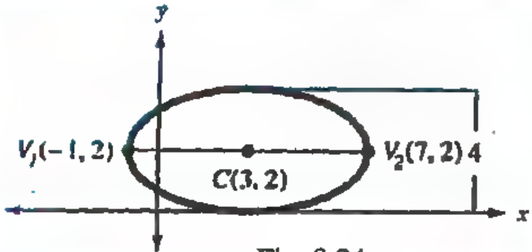


Fig. 8.24

The distance from the center C(3,2) to either vertex is a=4 units. The semiminor axis has a length of b=2. From the figure (8.24), we see that the major axis is parallel to the x–axis. The horizontal standard form of the ellipse (44) is used to obtain the required ellipse equation:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-3)^2}{16}+\frac{(y-2)^2}{4}=1,\quad C(h,k)=(3,2), a=4; b=2 \qquad (48)$$

8.2.2 Equations of tangent and normal

i) Recognize tangent and normal to an ellipse

A line which intersects an ellipse in two coincident points is a tangent. As in the case of the circle, but unlike that of parabola, there will be two tangents to an ellipse with a given slope. The formulation for tangents to an ellipse will be discussed in the succeeding sections.

ii) Point of intersection of an ellipse with a line including the condition of tangency

The given line and ellipse

$$y=mx+c \qquad (49)$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (50)$$

develops a system of two equations:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (51)$$
$$y=mx+c$$

The solution set $\{x, y\}$ of a system of equations (51) exists only, if the curves of the system (51) are intersecting. That set of points of intersection $\{x, y\}$ (a solution set) can be found by solving the nonlinear system (51) simultaneously.

The line (49) is used in ellipse (50) to obtain the quadratic equation in x:

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$
$$x^2(a^2m^2+b^2)+2a^2mcx+a^2(c^2-b^2)=0 \qquad (52)$$

The equation (52) being a quadratic equation in x, gives a set of two values x_1 and x_2 of x, which will be used in a line (49) to obtain a set of two values y_1 and y_2 of y.

Thus, a solution set $\{(x_1, y_1),(x_2, y_2)\}$ of the system (51) is of course a set of points of intersection of the system (51).

The points of intersection of the system (51) are real, coincident or imaginary, according as the roots of the quadratic equation (52) are real, coincident or imaginary; according as the discriminant of the quadratic equation (52):

$$Disc=4a^4m^2c^2-4(a^2m^2+b^2)(a^2)(c^2-b^2)>0, \text{ real and different}$$
$$Disc=4a^4m^2c^2-4(a^2m^2+b^2)(a^2)(c^2-b^2)=0, \text{ real and coincident}$$
$$Disc=4a^4m^2c^2-4(a^2m^2+b^2))(a^2)(c^2-b^2)<0, \text{ imaginary}$$

Example 8.2.6: [Points of Intersections]: Find the points of intersection of the line $2x-y-2=0$ and the ellipse $4x^2+9y^2=36$

Solution: The equations of the line and ellipse are:

$$2x-y-2=0 \qquad (53)$$
$$y=2x-2$$
$$4x^2+9y^2=36 \qquad (54)$$

The line (53) is used in an ellipse (54) to obtain the x–coordinates of the points of intersection:

$$4x^2+9(2x-2)^2=36$$

$$4x^2+9(4x^2+4-8x)-36=0$$

$$40x^2-72x=0 \Rightarrow x(40x-72)=0 \Rightarrow x=0,9/5$$

The x–coordinates are used in the line (53) to obtain the y–coordinates:

x=0, 9/5 give y=–2, 8/5 .

Thus, the set of two points of intersection (0,–2) and (9/5,8/5) are real and distint and the line 2x–y–2=0 intersects the ellipse (54) at points (0,–2) and (9/5,8/5).

iii) The equation of a tangent line in slope-form

If m is the slope of the tangent line to ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (55)$$

then the equation of that tangent line is of the form

$$y=mx+c \qquad (56)$$

Here c is to be calculated from the fact that the line (56) is tangent to ellipse (55).

The line (56) is used in ellipse (55) to obtain the quadratic equation in x:

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

$$x^2\left(a^2m^2+b^2\right)+2a^2mcx+a^2\left(c^2-b^2\right)=0 \qquad (57)$$

If the line (56) touches the ellipse (55), then the quadratic equation (57) has coincident roots for which the discriminant of the quadratic equation (57) equals zero:

$$4a^4m^2c^2-4\left(a^2m^2+b^2\right)\left(a^2\right)\left(c^2-b^2\right)=0$$

$$a^2m^2c^2-\left(a^2m^2+b^2\right)\left(c^2-b^2\right)=0, \textit{ divide out by } 4a^2$$

$$a^2m^2c^2-a^2m^2c^2+a^2m^2b^2-b^2c^2+b^4=0$$

$$a^2m^2b^2-b^2c^2+b^4=0$$

$$-b^2c^2=-\left(a^2m^2b^2+b^4\right)$$

$$c^2=a^2m^2+b^2$$

$$c=\pm\sqrt{a^2m^2+b^2} \qquad (58)$$

The equation (58) is the **condition of tangency**. The value of c from equation (58) is used in the line (56) to obtain the required equation of the tangent line:

$$y = mx + c = mx \pm \sqrt{a^2m^2 + b^2} \quad (59)$$

It is important to note that

- the equation of any tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the slope–form is:

$$y = mx \pm \sqrt{a^2m^2 + b^2} \quad (60)$$

- **Condition of Tangency:** The line y=mx+c should touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ under condition:

$$c = \pm\sqrt{a^2m^2 + b^2} \quad (61)$$

Example 8.2.7:[Tangency Condition]: For what value of c, the line 2x–y+c=0 will touch an ellipse $\frac{x^2}{3} + \frac{y^2}{4} = 1$. Use those values of c to find the tangent lines that should touch the given ellipse.

Solution: The values of c at which the line 2x–y+c=0 will touch the given ellipse through result (61) are:

$$c = \pm\sqrt{a^2m^2 + b^2} = \pm\sqrt{3(2)^2 + 4} = \pm 4$$

Here m=2 is the slope of the line 2x–y+c=0.

The required tangent lines that should touch the ellipse through result (60) is:

$$y = mx + c = 2x \pm 4, \; m = 2$$

iv) ***The equation of a tangent line to ellipse at a point***

The equation of a tangent line at a point $P(x_1, y_1)$ to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is:

$$y - y_1 = m_1(x - x_1) \quad (62)$$

Here m_1 is the slope of the tangent line to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a point $P(x_1, y_1)$ that can be found by differentiating $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to x:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-b^2x_1}{a^2y_1}=m_1, say \qquad (63)$$

The substitution of (63) in (62) is giving the equation of the tangent line at a point $P(x_1, y_1)$ to ellipse:

$$y-y_1=m_1(x-x_1)$$

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=-\frac{xx_1}{a^2}+\frac{x_1^2}{a^2}$$

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1, \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1\ at\ (x_1,y_1) \qquad (64)$$

Example 8.2.8:[Tangent Equation]: Find the equation of the tangent at a point P(3,12/5) to ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1.$

Solution: Result (64) is used to obtain the tangent line to the given ellipse:

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

$$\frac{x(3)}{25}+\frac{y(12/5)}{9}=1,\ (x_1,y_1)=(3,12/5), a^2=25, b^2=9$$

$$\frac{3x}{25}+\frac{12y}{45}=1 \Rightarrow 27x+60y-225=0 \Rightarrow 9x+20y-75=0$$

v) The equation of a normal line to ellipse at a point

The equation of a normal at a point $P(x_1, y_1)$ to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is:

$$y-y_1=m_2(x-x_1) \qquad (65)$$

Here m_2 is the slope of the normal to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ that can be found by differentiating $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to x:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-b^2x_1}{a^2y_1}=m_1$$

$$m_2=-\frac{1}{m_1}=\frac{a^2y_1}{b^2x_1},\ say \qquad (66)$$

The substitution of (66) in (65) is giving the normal equation at a point $P(x_1, y_1)$ to ellipse:

$$y-y_1=m_2(x-x_1)$$

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1),\ m_2=a^2y_1/b^2x_1$$

$$\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{x-x_1}{\frac{x}{a^2}} \qquad (67)$$

Example 8.2.9:[Normal Equation]: Find the normal equation at a point P(3,12/5) to ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$.

Solution: Result (67) is used to obtain the normal line to the given ellipse:

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}},\quad (x_1, y_1)=(3,12/5), a^2=25, b^2=9$$

$$\frac{x-3}{\frac{3}{25}}=\frac{y-\frac{12}{5}}{\left(\frac{12}{5}\right)/9},$$

$$\frac{25(x-3)}{3}=\frac{3(5y-12)}{4}$$

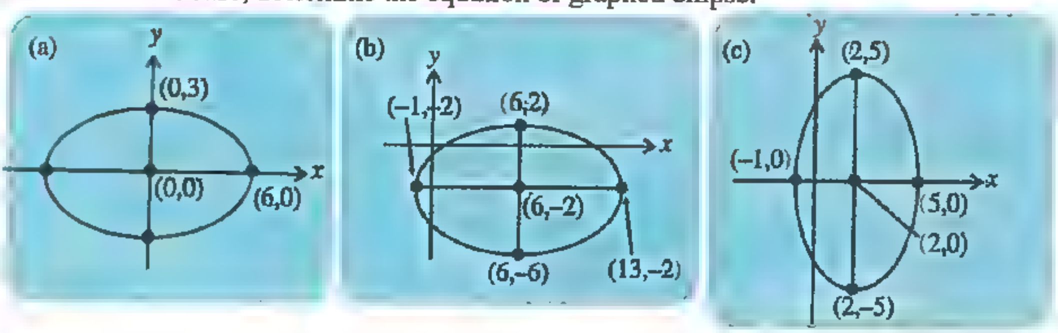
$$100(x-3)=9(5y-12)\Rightarrow 100x-300-45y+108=0\Rightarrow 100x-45y-192=0$$

Exercise 8.2

1. In each case, sketch the ellipse represented by the equation. Indicate the center, foci, endpoints of the major axis and end points of the minor axis:

   a. $\frac{x^2}{9}+\frac{y^2}{4}=1$ b. $\frac{x^2}{16}+\frac{y^2}{25}=1$ c. $\frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=1$

2. In each case, determine the equation of graphed ellipse:



3. In each case, write the equation of ellipse through the given information:

   a. Center is at (–3, 2), a=2, b=1, major axis is horizontal.

   b. Vertices are at (4,2) and (12,2), b=2.

   c. A focus is at (–2, 3), a vertex is at (6,3), length of minor axis is 6.

   d. Vertices are at (0,8) and (0,2), $c=\sqrt{5}$.

4. The shape of an ellipse depends on the eccentricity of the ellipse e=c/a. Determine

   a. the eccentricity of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.

   b. the equation of the ellipse with vertices are at (–5,0), and (5,0) and the eccentricity is e=3/5.

c. the eccentricity of the ellipse, if the length of the semimajor axis is a=4 and the length of the semiminor axis is b=2.

5. For what value of c,

a. the line $x-y+c=0$ will touch the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$?

b. the line $2x-y+c=0$ will touch the ellipse $\frac{x^2}{3}+\frac{y^2}{4}=1$?

c. the line $x+y+c=0$ will touch the ellipse $\frac{x^2}{25}+\frac{y^2}{11}=1$?

6. In each case, find the tangent equation and normal equation

a. at a point $(\frac{2\sqrt{5}}{3}, 2)$ to ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$?

b. at a point (0,2) to ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$?

c. at a point $(\sqrt{3}, \frac{1}{2})$ to ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$?

7. Find the tangent equation

a. to ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ which is perpendicular to the line $9x+8y-36=0$.

b. to ellipse $\frac{x^2}{7}+\frac{y^2}{4}=1$ which is parallel to the line $6x+21y-14=0$.

8.3 Hyperbola

The last of the conic sections to be considered has a definition similar to that of the ellipse. In the previous section, the ellipse was expressed in terms of the **sum** of two distances being a constant. Exactly, the hyperbola is expressed in terms of the **difference** of two distances being a constant.

i) Definition of hyperbola and its center, foci, vertices, eccentricity, focal chord, transverse and conjugate axes

Definition 8.3.1:[Hyperbola]: A hyperbola is the set of all points in the plane such that the difference of the distances from two fixed points (foci) to a point on the hyperbola hold off constant quantity.

The two fixed points $F_1(-c,0)$ and $F_2(c,0)$ are called the **foci** that lie on the x–axis at a distance c on each side of the origin. The hyperbola crosses the x–axis at two points $V_1(-a,0)$ and $V_2(a,0)$ that are at a distance **a** on each side of the origin called **vertices**. The **transverse axis** (major axis) of the hyperbola coincides with the x–axis, while the **conjugate axis** (minor axis) of the hyperbola coincides with the y–axis. This is shown in the fig. 8.25.

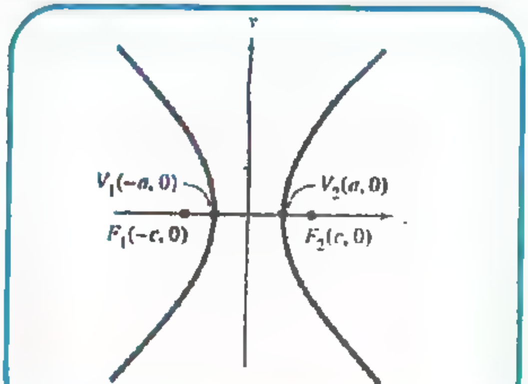


Fig. 8.25

To check the definition, the absolute value of the difference of the distances from the two foci $F_1(-c,0)$, $F_2(c,0)$ to the point $V_2(a,0)= P$, *say*, is:

$$|d(F_1,V_2)-d(F_2,V_2)|=|(a+c,0)-(c-a,0)|=|2a| \qquad (68)$$

Since a is a measured distance and is always positive, the constant specified in the definition of a hyperbola equals 2a.

ii) *Standard form of the hyperbola*

If P(x, y) is any point on the hyperbola, then the distances from the two foci $F_1(-c,0)$ and $F_2(c,0)$ to the point P(x, y) are the following:

$$F_1P=(x+c, y-0)$$

$$d(F_1,P)=\sqrt{(x+c)^2+(y)^2}$$

$$F_2P=(x-c, y-0)$$

$$d(F_2,P)=\sqrt{(x-c)^2+(y)^2}$$

The definition of hyperbola is used to obtain the general form of hyperbola:

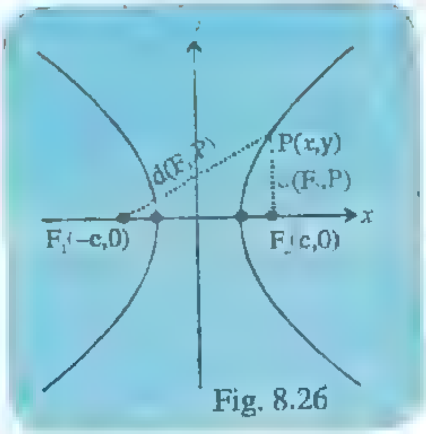


Fig. 8.26

$$d(F_1,P)-d(F_2,P)=2a$$

$$\sqrt{(x+c)^2+(y)^2}-\sqrt{(x-c)^2+y^2}=2a$$

$$\sqrt{(x+c)^2+y^2}=2a+\sqrt{(x-c)^2+y^2}$$

Squaring both sides to obtain

$$(x+c)^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$4a\sqrt{(x-c)^2+y^2}=4cx-4a^2$$

$$a\sqrt{(x-c)^2+y^2}=cx-a^2$$

Again squaring to obtain

$$a^2[(x-c)^2+y^2]=a^4-2a^2cx+c^2x^2$$

$$a^2\left(x^2-2cx+c^2+y^2\right)=a^4-2a^2cx+c^2x^2$$

$$a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2$$

$$a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$$

$$-(c^2-a^2)x^2+a^2y^2=-a^2(c^2-a^2)$$

$$-b^2x^2+a^2y^2=-a^2b^2,$$

$$b^2x^2-a^2y^2=a^2b^2, \qquad c^2=a^2+b^2$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad \text{divide out by } a^2b^2 \qquad (69)$$

Notice that $c^2 = a^2 - b^2$ for ellipse and that $c^2 = a^2 + b^2$ for the hyperbola. For ellipse, it is necessary that $a^2 > b^2$, but for the hyperbola, there is no restriction on the relative sizes for a and b (but c is still greater than a for the hyperbola).

8.3.1 Standard form of the equation of hyperbola

Definition 8.3.2:[Standard Form of Horizontal Hyperbola]: The standard form of the equation of a hyperbola with center at the origin and the x–axis as the transverse axis is shown in the figure (8.27):

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (70)$$

If we replace the foci on the y–axis, center at the origin, and pick any point P(x,y) on the plane, then we can develop the equation of the vertical ellipse given in definition 8.3.3.

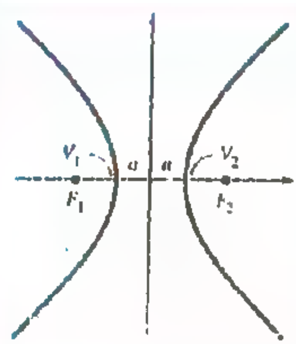


Fig. 8.27

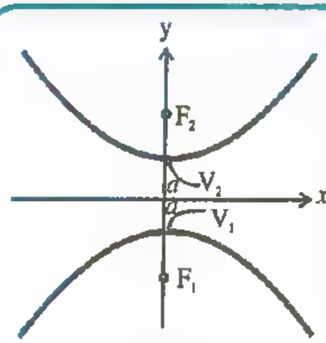


Fig. 8.28

Definition 8.3.3:[Standard Form of Vertical Hyperbola]: The standard form of the equation of a hyperbola with center at the origin and the y–axis as the transverse axis is shown in the figure (8.28):

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad (71)$$

Graphing Standard form Hyperbola: As with the other conics, we shall sketch a hyperbola by determining some information about the curve directly from the equation

by inspection. The end points of the transverse axis are the vertices of the hyperbola at $V_1(-a,0)$ and $V_2(a,0)$. The axis has a length of 2a. The conjugate axis coincides with the y–axis and has its end points at (0,b) and (0,–b). If we set x–intercept x=0, then there is no real number y–intercept (but $y=\pm\sqrt{-b^2}$ is in complex conjugate). The question is, why should we be concerned about the conjugate axis or the length b? The significance of b is determined by solving the standard form of hyperbola for y:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$-\frac{y^2}{b^2}=-\frac{x^2}{a^2}+1$$

$$y^2=b^2\left(\frac{x^2-a^2}{a^2}\right)$$

$$=\frac{b^2}{a^2}(x^2-a^2)=\frac{b^2x^2}{a^2}\left(1-\frac{a^2}{x^2}\right)$$

$$y=\pm\frac{bx}{a}\sqrt{1-\frac{a^2}{x^2}} \qquad (72)$$

Let us examine the fraction $\frac{a^2}{x^2}$ (a is constant). If we substitute larger and larger values for x, then the fraction a^2/x^2 becomes smaller and smaller. In fact, the fraction eventually gets very close to zero. Thus, for large values of x, the term $1-\frac{a^2}{x^2}$ approaches 1. Therefore, for large values of x, the y values approaches the value $\pm\frac{b}{a}x$, and the value of the hyperbola gets closer and closer to the lines:

$$y=\pm\frac{b}{a}x.$$

These lines are called the **asymptotes** of the hyperbola. As x takes on values that are at greater distances from the center of the hyperbola, the values of y (of the hyperbola) become closer and closer to the asymptotes even though they never actually reach the corresponding y–values of the asymptotes. Since these lines are easy to graph, the asymptotes are valuable aids in sketching the hyperbola.

- For horizontal standard form hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the asymptotes are the lines:

$$y=\pm\frac{b}{a}x \qquad (73)$$

- For vertical standard form hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, the asymptotes are the lines

$$y=\pm\frac{a}{b}x \qquad (74)$$

Definition 8.3.4:[Asymptotes]: If a straight line cuts a hyperbola in two points at an infinite distance from the origin and is itself at a finite distance from the origin is then called the **asymptotes.**

The orientation of the hyperbola with center C(0,0), vertices and foci are summarized in the box:

Orientation	Foci	Vertices
Horizontal	$F_1(-c,0), F_2(c,0)$	$V_1(-a,0), V_2(a,0)$
Vertical	$F_1(0,c), F_2(0,-c)$	$V_1(0,a), V_2(0,-a)$

Note that $c^2=a^2+b^2, c>0$.

Example 8.3.1:[Horizontal Hyperbola]: Sketch the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$, with center at the origin and the transverse axis is at the x–axis. Determine the vertices and foci of the hyperbola.

Solution: The given hyperbola equation is in the standard form of the equation of a hyperbola (70) with transverse axis along the x–axis. This tells us that $a^2=9, a=\pm3$ and $b^2=16, b=\pm4$. The vertices of the hyperbola are $V_1(-3,0)$ and $V_2(3,0)$. The value of c for foci can be found by using the formula:

$$c^2=a^2+b^2=9+16=25, c=\pm5$$

The foci are therefore $F_1(-5,0)$ and $F_2(5,0)$. The asymptotes are the lines

$$y=\frac{b}{a}x=\frac{4}{3}x, \quad y=-\frac{b}{a}x=-\frac{4}{3}x$$

For sketching the hyperbola, the end points of the conjugate axis (0,–4) and (0,4) are located, then draw the lines through the points (0,–4) and (0,4) parallel to the

x–axis. Similarly, draw the lines through the end points of the transverse axis $V_1(-3,0)$ and $V_2(3,0)$ parallel to y–axis to complete the rectangle. The resultant rectangle and the extended diagonals of the rectangle are the asymptotes of the hyperbola. The sketch of the hyperbola is shown in the Fig. 8.29:

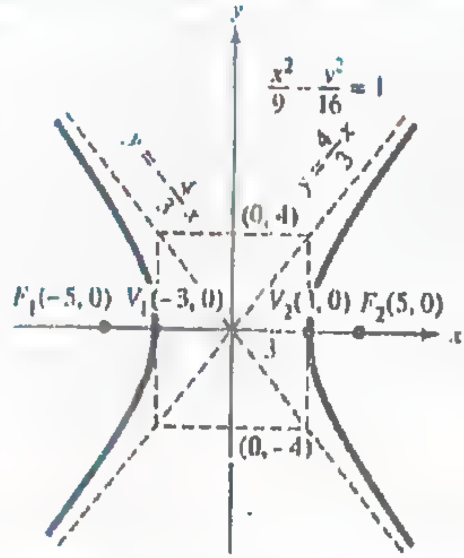


Fig. 8.29

Example 8.3.2:[Vertical Hyperbola]: Sketch the hyperbola $16y^2-9x^2=144$, with center at the origin and the transverse axis is at the y–axis. Determine the vertices and foci of the hyperbola.

Solution: Rewrite the given hyperbola in the standard form of the hyperbola (71):

$$16y^2-9x^2=144$$

$$\frac{16y^2}{144}-\frac{9x^2}{144}=1 \Rightarrow \frac{y^2}{9}-\frac{x^2}{16}=1 \qquad (75)$$

If the transverse axis is along the y–axis, then select $a^2=9, a=\pm3$ and $b^2=16, b=\pm4$. The vertices of the hyperbola are $V_1(0,3)$ and $V_2(0,-3)$. The end points of the conjugate axis are (–4,0) and (4,0). The value of c for foci can be found by using the formula:

$$c^2=a^2+b^2=9+16=25, c=\pm5$$

The foci are therefore $F_1(0,5)$ and $F_2(0,-5)$. The asymptotes are the lines

$$y=\frac{a}{b}x=\frac{3}{4}x, \quad y=-\frac{a}{b}x=-\frac{3}{4}x$$

Sketch the rectangle formed by the points $(0,\pm 3)$ and $(\pm 4,0)$ and then sketch the asymptotes using the diagonals of the rectangle. With the asymptotes, vertices and foci, it is easy to sketch the hyperbola, as shown in the Fig. 8.30.

i) Equation of hyperbola through its elements

Example 8.3.3:[Equation of Hyperbola]: Find the vertices, foci, eccentricity and the asymptotes of the hyperbola $16x^2-9y^2=144$.

Solution: Rewrite the given hyperbola in the standard form :

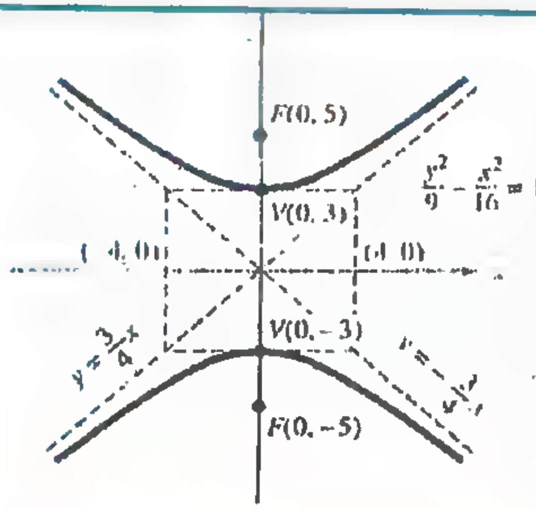


Fig. 8.30

$$16x^2-9y^2=144$$

$$\frac{16x^2}{144}-\frac{9y^2}{144}=1 \Rightarrow \frac{x^2}{9}-\frac{y^2}{16}=1$$

If the transverse axis is the x–axis, then select $a^2=9, a=3$ and $b^2=16, b=4$. The value of c is obtained by formula $c=\sqrt{a^2+b^2}=\sqrt{9+16}=\pm 5$.

The vertices, foci, eccentricity and asymptotes of the given hyperbola are the following:

$V_1(a,0)=V_1(3,0), V_2(-a,0)=V_2(-3,0)$ vertices

$F_1(c,0)=F_1(5,0), F_2(-c,0)=F_2(-5,0)$ foci

$e=\frac{c}{a}=\frac{5}{3}>1$ eccentricity

$y=\pm\frac{b}{a}x=\pm\frac{4}{3}x$ asymptotes

Example 8.3.4:[Equation of Hyperbola]: Find the equation of hyperbola, when one focus is at $(0,6)$, center is at C(0,0) and the eccentricity is 3.

Solution: Focus$(0,6)$ gives c=6. This indicates that the transverse axis is the y–axis. The eccentricity 3 is giving the value of a:

$$e=\frac{c}{a}=>3=\frac{6}{a}=>a=2$$

These values of a and c is used in the formula to obtain the value of b:

$$c^2=a^2+b^2$$

$$b^2=c^2-a^2=36-4=32$$

The required hyperbola equation is:

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

$$\frac{y^2}{4}-\frac{x^2}{32}=1,\ \ a^2=4, b^2=32$$

ii) Conversion of an equation to the standard form equation of hyperbola

If any point (h,k) on the plane is selected as the center of the hyperbola and a major axis parallel to the x–axis or y–axis is selected, then with the geometrical definition, a new set of equations for hyperbola can be derived through translation of axes.

Definition 8.3.5:[Translation of Hyperbola Horizontally]: The standard form of the equation of a hyperbola with center at C(h,k), vertices at $V_1(h+a,k)$ and $V_2(h-a,k)$, foci at $F_1(h-c,k)$ and $F_2(h+c,k)$ is:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \qquad (76)$$

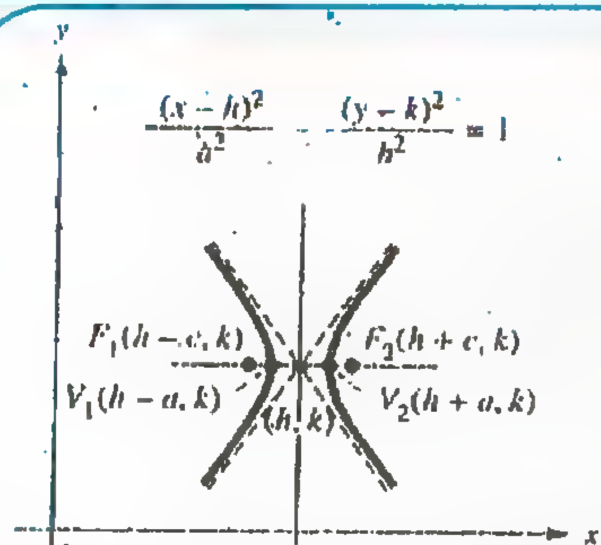


Fig. 8.31

Definition 8.3.6:[Translation of Hyperbola Vertically]: The standard form of the equation of a hyperbola with center at C(h,k), vertices at $V_1(h,k+a)$ and $V_2(h,k-a)$, foci at $F_1(h,k+c)$ and $F_2(h,k-c)$ is:

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1 \qquad (77)$$

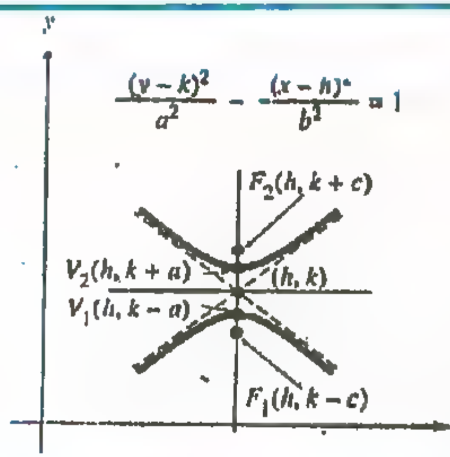


Fig. 8.32

The orientations of the hyperbola equation with center C(h,k) are summarized in the box:

Orientation	Foci	Vertices
Horizontal	$F_1(h-c,k), F_2(h+c,k)$	$V_1(h-a,k), V_2(h+a,k)$
Vertical	$F_1(h,k+c), F_2(h,k-c)$	$V_1(h,k+a), V_2(h,k-a)$

Note that $c^2=a^2+b^2$, c>0.

Example 8.3.6:[Translation of Hyperbola]: Sketch the hyperbola

$$\frac{(x-1)^2}{144}-\frac{(y+2)^2}{25}=1$$

with center at C(h, k) and the transverse axis is the x–axis. Determine the vertices and foci of the hyperbola.

Solution: The given hyperbola is:

$$\frac{(x-1)^2}{144}-\frac{(y+2)^2}{25}=1 \qquad (78)$$

The given hyperbola (78) with substitution

X=x–h=x–1 and Y=y–k=y+2, h=1, k=–2 gives the new hyperbola in the XY–system:

$$\frac{X^2}{144}-\frac{Y^2}{25}=1 \qquad (79)$$

The hyperbola (79) has a transverse axis on the x–axis that give $a^2=144, a=12$ and $b^2=25, b=5$. The value of c is obtained by formula:

$c=\sqrt{a^2+b^2}=\sqrt{144+25}=13$

The coordinates of the vertices and foci of the new hyperbola in the XY–system are the following:

$V_1(-a,0)=V_1(-12,0), V_2(a,0)=V_2(12,0)$ vertices

$F_1(-c,0)=F_1(-13,0), F_2(c,0)=F_2(13,0)$ foci

The asymptotes in the XY–system are the lines:

$$Y=\frac{b}{a}X=\frac{5}{12}X, \qquad Y=-\frac{b}{a}X=-\frac{5}{12}X$$

The coordinates of the vertices and foci of the given hyperbola (78) in the xy–system are the following:

$V_1(h+a,k)=V_1(13,-2), V_2(h-a,k)=V_2(-11,-2)$ vertices

$F_1(h-c,k)=F_1(-12,-2), F_2(h+c,k)=V_2(14,-2)$ foci

The asymptotes in the XY–system can be converted in the xy–system to obtain:

$$Y=\frac{b}{a}X$$

$$=\frac{5}{12}X,\ a=12, b=5$$

$$(y-k)=\frac{5}{12}(x-h),\ X=x-h, Y=y-k,\ h=1, k=-2$$

$$y+2=\frac{5}{12}(x-1),$$

$$12y+24=5x-5 \Rightarrow 12y-5x+29=0$$

$$Y=-\frac{b}{a}X$$

$$=-\frac{5}{12}X$$

$$(y-k)=-\frac{5}{12}(x-h)$$

$$y+2=-\frac{5}{12}(x-1) \Rightarrow 12y+24=-5x+5 \Rightarrow 12y+5x+19=0$$

Having sketched the asymptotes and the vertices, we can sketch the hyperbola as shown in the figure (8.33):

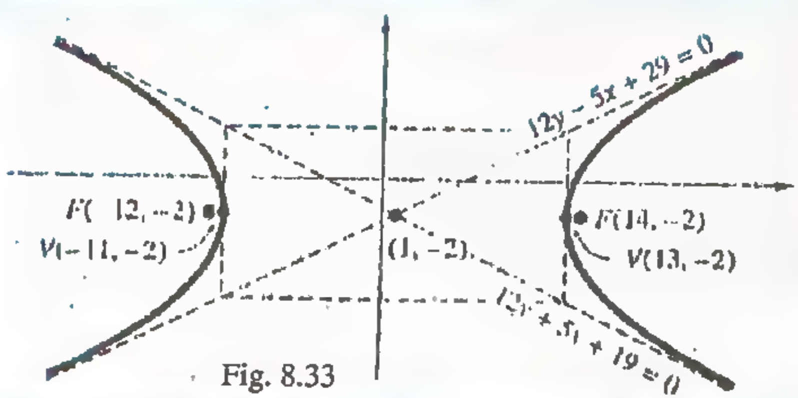


Fig. 8.33

- For translating the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ horizontally, the asymptotes are the lines:

$$(y-k)=\pm\frac{b}{a}(x-h) \qquad (80)$$

- For translating the hyperbola $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ vertically, the asymptotes are the lines:

$$(y-k)=\pm\frac{a}{b}(x-h) \qquad (81)$$

8.3.2 Equations of tangent and normal

i) Recognize tangent and normal to hyperbola

A line which intersects a hyperbola in two coincident points is a tangent. For the hyperbola, there will be two tangents [real and distanct, coincident (with an asymptote), or complex] with a given slope. The formulation for tangents to hyperbola will be discussed in the succeeding sections.

ii) Point of intersection of hyperbola with a line including the condition of tangency

The given line and hyperbola

$$y=mx+c \qquad (82)$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (83)$$

develops a system of two equations:

$$\left.\begin{array}{l}\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\\ y=mx+c\end{array}\right. \qquad (84)$$

The solution set $\{x, y\}$ of a system of equations (84) exists only, if the curves of the system (84) are intersecting. That set of points of intersection $\{x, y\}$ (a solution set) can be found by solving the system (84) simultaneously.

The line (82) is used in hyperbola (83) to obtain the quadratic equation in x:

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

$$x^2\left(b^2-a^2m^2\right)-2a^2mcx-a^2\left(c^2+b^2\right)=0 \qquad (85)$$

The equation (85) being a quadratic equation in x, gives a set of two values x_1 and x_2 of x, which will be used in a line (82) to obtain a set of two values y_1 and y_2 of y.

The solution set $\{(x_1,y_1),(x_2,y_2)\}$ of the system (84) is of course a set of points of intersection of the system (84).

The points of intersection of the system (84) are real, coincident or imaginary, according as the roots of the quadratic equation (85) are real, coincident or imaginary, or according as the discriminant of the quadratic equation (85):

$$Disc=4a^4m^2c^2+4\left(b^2-a^2m^2\right)\left(a^2\right)\left(c^2+b^2\right)>0, \text{ real and different}$$

$$Disc=4a^4m^2c^2+4\left(b^2-a^2m^2\right)\left(a^2\right)\left(c^2+b^2\right)=0, \text{ real and coincident}$$

$$Disc=4a^4m^2c^2+4\left(b^2-a^2m^2\right)\left(a^2\right)\left(c^2+b^2\right)<0, \text{ imaginary}$$

Example 8.3.7:[Point of intersection]: Find the points of intersection of the line x–y–1=0 and the hyperbola $4x^2-y^2=4$.

Solution: The equations of the line and hyperbola are :

$$x-y-1=0$$
$$y=x-1 \qquad (86)$$
$$4x^2-y^2=4 \qquad (87)$$

The line (86) is used in hyperbola (87) to obtain the x–coordinates of the points of intersection:

$$4x^2-(x-1)^2=4$$

$$4x^2-(x^2+1-2x)-4=0 \Rightarrow 3x^2+2x-5=0 \Rightarrow x=1,-\frac{5}{3}$$

The x–coordinates are used in the line (86) to obtain the y–coordinates:

x=1, –5/3 give y=0, –8/3

Thus, the set of two points of intersection (1,0) and (–5/3,–8/3) are real and distant and the line x–y–1=0 intersects the hyperbola (87) at points (1,0) and (–5/3,–8/3).

iii) *The equation of a tangent line in slope-form*

If m is the slope of the tangent line to hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (88)$$

then the equation of that tangent line is of the form

$$y=mx+c \qquad (89)$$

Here c is to be calculated from the fact that the line (89) is tangent to hyperbola (88).

The line (89) is used in hyperbola (88) to obtain the quadratic equation in x:

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

$$x^2\left(b^2-a^2m^2\right)-2a^2mcx-a^2\left(c^2+b^2\right)=0 \qquad (90)$$

If the line (89) touches the hyperbola (88), then the quadratic equation (90) has coincident roots for which the discriminant of the quadratic equation (90) is going to be zero:

$$4a^4m^2c^2+4\left(b^2-a^2m^2\right)\left(a^2\right)\left(c^2+b^2\right)=0$$

$$a^2m^2c^2+\left(b^2-a^2m^2\right)\left(c^2+b^2\right)=0, \text{ divide out by } 4a^2$$

$$a^2m^2c^2-a^2m^2c^2-a^2m^2b^2+b^2c^2+b^4=0$$

$$-a^2m^2b^2+b^2c^2+b^4=0$$

$$-a^2m^2+c^2+b^2=0$$

$$c^2=(a^2m^2-b^2)$$

$$c^2=a^2m^2-b^2$$

$$c=\pm\sqrt{a^2m^2-b^2} \qquad (91)$$

The equation (91) is the **condition of tangency.** The value of c from equation (91) is used in the line (89) to obtain the required equation of the tangent line:

$$y=mx+c=mx\pm\sqrt{a^2m^2-b^2} \qquad (92)$$

It is important to note that

- the equation of any tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in the slope–form is:

$$y=mx\pm\sqrt{a^2m^2+b^2} \qquad (93)$$

- the line y=mx+c should touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ under condition:

$$c=\pm\sqrt{a^2m^2-b^2} \qquad (94)$$

Example 8.3.8:[Tangency Condition]: For what value of c, the line $y=\frac{5}{2}x+c$ will touch the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$. Use those values of c to find the tangent lines that should touch the given hyperbola.

Solution: The values of c at which the line $y=\frac{5}{2}x+c$ will touch the given hyperbola through result (91) are:

$$c=\pm\sqrt{a^2m^2-b^2}=\pm\sqrt{4(25/4)-9}=\pm 4,\quad a^2=4, b^2=9, m=5/2$$

Here m=5/2 is the slope of the line $y=\frac{5}{2}x+c$. The required tangent lines that should touch the hyperbola through result (92) is:

$$y=mx+c=\frac{5}{2}x\pm 4,\ m=5/2$$

iv) The equation of a tangent line to hyperbola at a point

The equation of the tangent at a point $P(x_1, y_1)$ to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is:

$$y-y_1=m_1(x-x_1) \qquad (95)$$

Here m_1 is the slope of the tangent line to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ that can be found by differentiating $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with respect to x:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{b^2x_1}{a^2y_1} = m_1, say \qquad (96)$$

The substitution of (96) in (95) is giving the equation of the tangent line at a point $P(x_1, y_1)$ to hyperbola:

$$y - y_1 = m_1(x - x_1)$$

$$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{x_1^2}{a^2}$$

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1, \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \; at \; (x_1, y_1) \qquad (97)$$

Example 8.3.9:[Tangent Equation]: Find the equation of the tangent at a point P(5,16/9) to hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1.$

Solution: Result (97) is used to obtain the tangent line to the given hyperbola:

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1, \quad a^2 = 9, b^2 = 16$$

$$\frac{x(5)}{9} - \frac{y(16/9)}{16} = 1, \quad (x_1, y_1) = (5, 16/9)$$

$$\frac{5x}{9} - \frac{16y}{144} = 1$$

$$80x - 16y = 144$$

$$80x - 16y - 144 = 0$$

$$5x - y - 9 = 0$$

v) *The equation of a normal line to hyperbola at a point*

The equation of the normal at a point $P(x_1, y_1)$ to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is:

$$y - y_1 = m_2(x - x_1) \qquad (98)$$

Here m_2 is the slope of the normal to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, at a point $P(x_1, y_1)$ that can be found by differentiating $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with respect to x:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{b^2x_1}{a^2y_1}=m_1,$$

$$m_2=-\frac{1}{m_1}=\frac{-a^2y_1}{b^2x_1}, say \qquad (99)$$

The substitution of (99) in (98) is giving the equation of normal at a point $P(x_1, y_1)$ to hyperbola:

$$y-y_1=m_2(x-x_1)$$

$$y-y_1=\frac{-a^2y_1}{b^2x_1}(x-x_1), \; m_2=-a^2y_1/b^2x_1$$

$$\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{-(x-x_1)}{\frac{x_1}{a^2}} \qquad (100)$$

Example 8.?.10:[Normal Equation]: Find the equation of normal at a point P(5,16/9) to hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$.

Solution: Result (100) is used to obtain the normal line to the given hyperbola:

$$\frac{-(x-x_1)}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}, \; a^2=9, b^2=16$$

$$\frac{-(x-5)}{\frac{5}{9}} = \frac{y-\frac{16}{9}}{\left(\frac{16}{9}\right)/16}, \quad (x_1, y_1) = (5, 16/9)$$

$$\frac{-9(x-5)}{5} = \frac{(9y-16)}{1}$$

Exercise 8.3

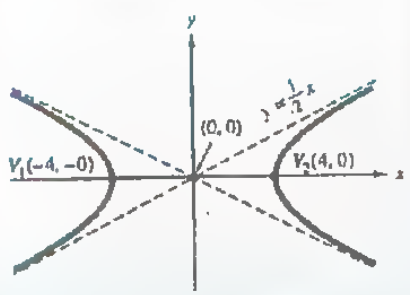
1. In each case, sketch the hyperbola represented by the equation. Indicate the center, vertices, foci and the equations of the asymptotes:

a. $\frac{x^2}{4} - \frac{y^2}{9} = 1$ b. $\frac{y^2}{25} - \frac{x^2}{4} = 1$

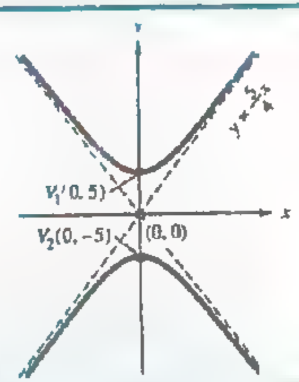
c. $\frac{(x-2)^2}{9} - \frac{(y-3)^2}{16} = 1$ d. $\frac{(y+1)^2}{16} - \frac{(x+3)^2}{25} = 1$

2. In each case, determine the equation of graphed ellipse.

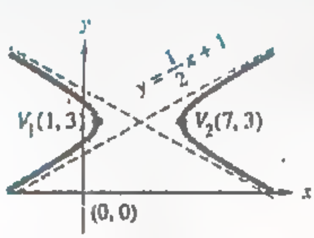
a.



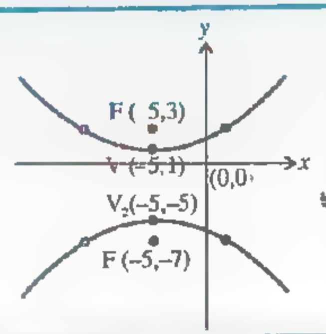
b.



c.



d.



3. In each case, write the equation of hyperbola through the given information:

a. Foci are at (0,3) and (0,–3), one vertex is at (0,–2).

b. Foci are at $(\sqrt{2},0)$ and $(-\sqrt{2},0)$, one vertex is at (1,0).

c. Vertices are at (5,0) and (–5,0), one focus is at (–7,0).

d. Vertices are at (3,0) and (–3,0), and asymptotes are the lines y=3x and y= –3x.

e. Transverse axis is the x–axis, asymptotes are the lines y=3x and y= –3x.

f. Transverse axis is the y–axis, asymptotes are the lines y=3x and y= –3x.

g. Foci are at (5,0) and (–5,0), eccentricity is 5/3.

h. Vertices are at (9,0) and (–9,0), whose asymptotes are perpendicular to each other.

4. Determine the path of a point that moves so that the difference of its distances from

a. the points (–5,0) and (5,0) is 8.

b. the points (0,–13) and (0,13) is 10.

5. Write the equation of the hyperbola

a. with vertices at (2,–2), (–4,–2) and that passes through the point with coordinates (5,1).

b. with vertices at (–3,1), (–3,3) and that passes through the point with coordinates (0,4).

6. In each case, sketch the rectangular hyperbola and identify the vertices, the foci and the asymptotes:

a. $(x+1)^2-(y-2)^2=1$ b. $\dfrac{(x-3)^2}{4}-\dfrac{(y+1)^2}{4}=1$

7. In each case, find the points of intersection in between the line and the hyperbola:

a. $xy=4,\quad y=x-3$ b. $x^2-y^2=1,\quad y=\sqrt{3}(x-1)$

c. $\dfrac{y^2}{1}-\dfrac{x^2}{4}=1,\quad x+4y-4=0$ d. $\dfrac{y^2}{9}-\dfrac{x^2}{4}=1,\quad y=3(x+1)$

8. For what value of c,

a. the line y=x+c will touch the hyperbola $\dfrac{x^2}{16}-\dfrac{y^2}{4}=1$?

b. the line y= –x+c will touch the hyperbola $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$?

c. the line $y=x+c$ will touch the hyperbola $\frac{x^2}{4}-\frac{y^2}{1}=1$?

9. In each case, find the tangent line equation and normal line equation

a. at a point $\left(-\sqrt{13},\frac{9}{2}\right)$ to hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$?

b. at a point $\left(\frac{16}{3},5\right)$ to hyperbola $\frac{y^2}{9}-\frac{x^2}{16}=1$?

8.4 Translation and Rotation of Axes

If the coordinates of a point or the equation of a curve be given with reference to a system of axes, rectangular or oblique, then the coordinates of the same point or the equation of the same curve can be obtained with reference to another system of axes, rectangular or oblique. The process of so changing the coordinates of a point or the equation of a curve is called the **transformation of coordinates.**

i) Translation and rotation of axes

In general, we come across to define three types of change of axes that are the following:

1. **Translation of Axes:** This will be used in changing the origin only and the new axes are parallel to the old ones.
2. **Rotation of Axes:** This will be used in changing the directions of the axes without changing the origin of the system.
3. **General Transformation:** This will be used, when the change of the direction and the origin of the axes both come together.

Definition:8.4.1: [Translation of Axes]: The relationship between the two sets of coordinate axes is called the **translation of axes.**

Definition:8.4.2:[Rotation of Axes]: The rotational relationship between the two sets of coordinate axes is called the **rotation of axes.**

ii) Equations of transformation for translation of axes

If O(0,0) is the old origin of the set of old rectangular coordinate axes ox and oy, then the coordinates of a point P with respect to the old axes are P(x, y).

If O(h, k) is the new origin of the set of new rectangular coordinate axes OX and OY parallel to the old rectangular coordinate axes, then the coordinates of a point P with respect to the new axes are P(X, Y).

If PM and ON are perpendicular to old coordinate axis ox, where PM intersects the new coordinate axis OX at M_1, then, the following assumptions

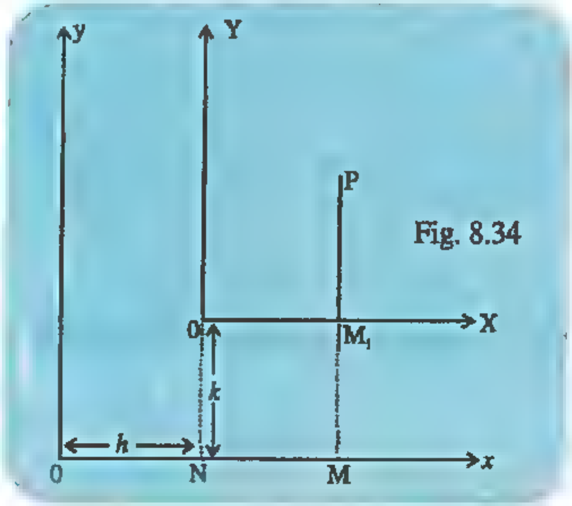


Fig. 8.34

$ON = h,\ NO = k, OM = x,\ MP = y$ also $OM_1 = X,\ M_1P = Y$

through old rectangular coordinate axes

$$x = OM = ON + NM = ON + OM_1 = h + X$$

$$y = MP = MM_1 + M_1P = NO + M_1P = k + Y$$

develops a set of rectangular coordinate axes in terms of new coordinates X, Y by means of the relation:

$$x = X + h,\quad y = Y + k \qquad (101)$$

The set of equations (101) are the equations of **transformation for translation of axes**. By making this substitution in a given equation, a new equation of the same graph is obtained, referred now to the new translated axes.

iii) Equations of transformation for rotation of axes

If ox and oy is the set of old rectangular coordinate axes, then the set of new rectangular coordinate axes OX and OY is obtained by rotating the old rectangular coordinates through an angle $\theta, 0 < \theta < 90^0$ in anti-clockwise direction.

If the coordinates of a point P with respect to the old axes are P(x, y), then the coordinates of a point P with respect to the new axes are P(X, Y).

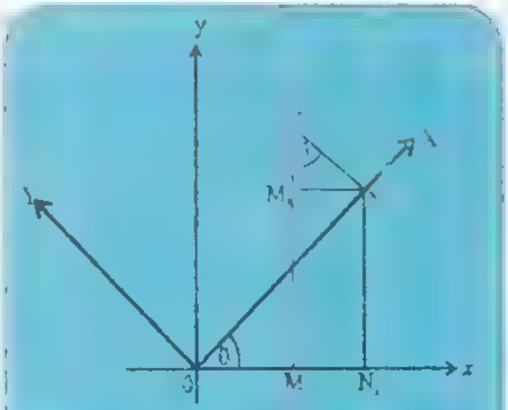


Fig. 8.35

If PM and PN are perpendicular to ox and OX and NN_1 and NM_1 are perpendicular to ox and PM, then, the following assumptions

$oM = x,\ MP = y,\ ON = X,\ NP = Y,\ \angle M_1PN = \theta$

through old rectangular coordinate axes

$x = oM$

$= oN_1 - MN_1 = oN_1 - M_1N = ON\cos\theta - NP\sin\theta = X\cos\theta - Y\sin\theta$

$y = MP$

$= MM_1 + M_1P = N_1N + M_1P = ON\sin\theta + NP\cos\theta = X\sin\theta + Y\cos\theta$

develops a set of rectangular coordinate axes in terms of new coordinates X, Y by means of the relation:

$x = X\cos\theta - Y\sin\theta,\quad y = X\sin\theta + Y\cos\theta$ (102)

An equivalent form of the relations (102) is:

$X = x\cos\theta + y\sin\theta,\quad Y = -x\sin\theta + y\cos\theta$ (103)

Example 8.4.1: Translate to parallel axes through the point (1,-2) the conic $4x^2 + 25y^2 - 8x + 100y + 4 = 0$.

Solution: Substitute x=X+h=X+1 (h=1) and y=Y+k=Y-2 (k=-2) in the given equation

$$4x^2+25y^2-8x+100y+4=0$$

$$4(X+1)^2+25(Y-2)^2-8(X+1)+100(Y-2)+4=0$$

which yields the standard form of ellipse in XY-plane by completing the square:

$$\frac{X^2}{25}+\frac{Y^2}{4}=1,\ XY-System \qquad (104)$$

The standard form of the given conic (ellipse) equation in xy-plane is obtained by backward substitution of x= X+1 and y= Y-2:

$$\frac{(x-1)^2}{25}+\frac{(y+2)^2}{4}=1,\ X=x-1,\ Y=y+2,\ xy-system$$

Example 8.4.2: Transform to axes inclined at an angle 45^0 to the original axes of the conic $x^2+y^2-8x+4xy-1=0$.

Solution: The substitution of equation (102)

$$\begin{aligned} x &= X\cos\theta - Y\sin\theta,\ \theta=45^0 \\ &= X\cos 45 - Y\sin 45 \\ &= \frac{1}{2}X-\frac{\sqrt{2}}{2}Y \\ &= \frac{\sqrt{2}}{2}(X-Y),\ \sin 45=\cos 45=\frac{1}{2} \end{aligned}$$

$$\begin{aligned} y &= X\sin\theta + Y\cos\theta \\ &= X\cos 45 + Y\sin 45 \\ &= \frac{\sqrt{2}}{2}X+\frac{\sqrt{2}}{2}Y=\frac{\sqrt{2}}{2}(X+Y) \end{aligned}$$

in the given conic equation

$$\frac{2}{4}(X-Y)^2+\frac{2}{4}(X+Y)^2-8\frac{\sqrt{2}}{2}(X-Y)+4\left(\frac{2}{4}\right)(X-Y)(X+Y)-1=0$$

to obtain the transformed equation of hyperbola

$$3X^2-Y^2-4\sqrt{2}(X-Y)-1=0$$

which yields the given conic by substituting:

$$\left.\begin{aligned} X+Y &= \frac{2y}{\sqrt{2}} \\ X-Y &= \frac{2x}{\sqrt{2}} \end{aligned}\right\} \Rightarrow X=\frac{1}{\sqrt{2}}(x+y),\ Y=\frac{1}{\sqrt{2}}(x-y)$$

iv) New origin and new axes with respect to old origin and old axes

This is actually the general transformation (third type) which requires both translation and rotation of axes. The procedure developed is as under:

If o(0,0) is the old origin of the set of old rectangular coordinate axes ox and oy, then the origin of the set of rectangular coordinate axes OX and OY parallel to the old rectangular coordinate axes is O(h,k). Further, the set of new rectangular coordinate axes OX' and OY' is obtained by rotating the rectangular coordinates axes OX and OY through an angle $\theta, 0 < \theta < 90^0$ in anti-clockwise direction.

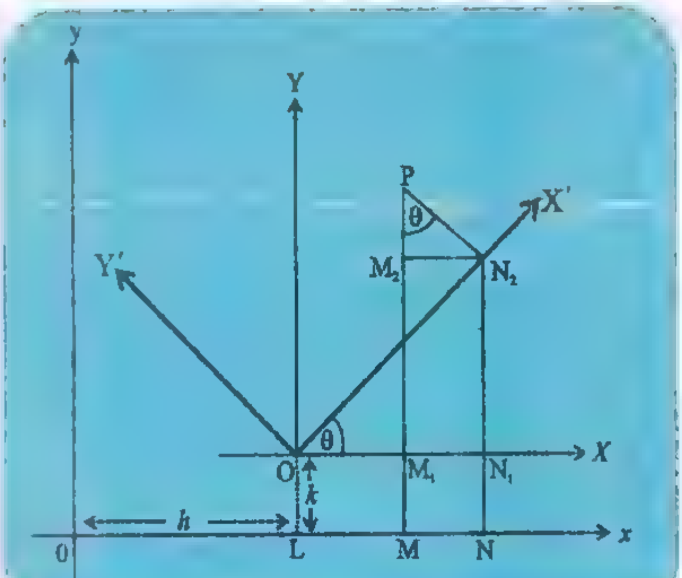


Fig. 8.36

If the rectangular coordinates of a point P with respect to old rectangular coordinate axes are P(x,y), then the rectangular coordinates of a point P with respect to rectangular coordinates axes OX, OY and new rectangular coordinates axes OX', OY' are respectively P(X,Y) and $P(X', Y')$.

The following assumptions

$$oM = x, MP = y, OM_1 = X, M_1P = Y, ON_2 = X', N_2P = Y'$$

through old rectangular coordinate axes

$$x = oM$$
$$= oN - MN$$
$$= oL + LN - MN$$
$$= oL + ON_1 - M_2N_2$$
$$= oL + ON_2\cos\theta - N_2P\sin\theta = h + X'\cos\theta - Y'\sin\theta$$
$$y = MP$$
$$= MM_1 + M_1M_2 + M_2P$$
$$= OL + N_1N_2 + M_2P = k + ON_2\sin\theta + N_2P\cos\theta$$
$$= k + X'\sin\theta + Y'\cos\theta$$

develops a set of rectangular coordinate axes in terms of new coordinates X' and Y' by means of the relation:

$$x = h + X'\cos\theta - Y'\sin\theta,\quad y = k + X'\sin\theta + Y'\cos\theta \qquad (105)$$

Example 8.4.3: Transform to new axes inclined at an angle 45^0 to the original axes of the conic $x^2 + y^2 - 8x + 4xy - 1 = 0$ through (2, 3).

Solution: The substitution of equation (105)

$$x = h + X'\cos\theta - Y'\sin\theta,\ \theta = 45^0$$
$$= h + X'\cos 45 - Y'\sin 45$$
$$= 2 + \frac{\sqrt{2}}{2}X' - \frac{\sqrt{2}}{2}Y'$$
$$= 2 + \frac{\sqrt{2}}{2}(X' - Y'),\ \sin 45 = \cos 45 = \frac{\sqrt{2}}{2}$$
$$y = k + X'\sin\theta + Y'\cos\theta$$
$$= k + X'\cos 45 + Y'\sin 45$$
$$= 3 + \frac{\sqrt{2}}{2}X' + \frac{\sqrt{2}}{2}Y' = 3 + \frac{\sqrt{2}}{2}(X' + Y')$$

in the given conic equation

$$\left[2 + \frac{2}{\sqrt{2}}(X' - Y')\right]^2 + \left[3 + \frac{2}{\sqrt{2}}(X' + Y')\right]^2 - 8\left[2 + \frac{\sqrt{2}}{2}(X' - Y')\right]$$
$$+ 4\left[2 + \frac{\sqrt{2}}{2}(X' - Y')\right]\left[3 + \frac{\sqrt{2}}{2}(X' + Y')\right] - 1 = 0$$

to obtain the transformed equation of hyperbola

$$3X'^2 + Y'^2 + 4\sqrt{2}(X'-Y') + 7\sqrt{2}(X'+Y') + 20 = 0$$

that yields the given conic by substituting:

$$\left.\begin{aligned} x &= 2+\frac{\sqrt{2}}{2}(X'-Y') \\ y &= 3+\frac{\sqrt{2}}{2}(X'+Y') \end{aligned}\right\} \Rightarrow X' = \frac{(x-2)}{\sqrt{2}} + \frac{(y-3)}{\sqrt{2}},\ Y' = \frac{(y-3)}{\sqrt{2}} - \frac{(x-2)}{\sqrt{2}}$$

- ***Angle through which the axes be rotated about the origin so that the product term xy is removed from the transformed equation***

The substitution of the equations of transformation (102)

$$x = X\cos\theta - Y\sin\theta,\quad y = X\sin\theta + Y\cos\theta$$

in the conic of the form $ax^2 + 2hxy + by^2$ is

$$\begin{aligned} ax^2 + 2hxy + by^2 &= a(X\cos\theta - Y\sin\theta)^2 \\ &\quad + 2h(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + b(X\sin\theta + Y\cos\theta)^2 \\ &= \left(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta\right)X^2 \\ &\quad + \left\{-2(a-b)\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta)\right\}XY \\ &\quad + \left\{a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta\right\}Y^2 \end{aligned}$$

The expression $ax^2 + 2hxy + by^2$ will be of the form $aX^2 + bY^2$, if the coefficient of XY term on the right side of the above equation equals zero:

$$\left\{-2(a-b)\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta)\right\} = 0$$

$$-(a-b)\sin 2\theta + 2h\cos 2\theta = 0$$

$$-(a-b)\sin 2\theta = -2h\cos 2\theta$$

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{2h}{a-b}$$

$$\tan 2\theta = \frac{2h}{a-b}$$

$$\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b} \qquad (106)$$

Example 8.4.4: At what angle the axes are rotated about the origin so that the transformed equation of the conic $9x^2+4y^2+12xy-x-y=0$ does not contain the term involving XY?

Solution: If the axes of the given conic are rotated through an angle θ, then the angle θ can be found through result (106):

$$\theta=\frac{1}{2}\tan^{-1}\frac{2h}{a-b},\quad a=9, b=4, h=6$$

$$=\frac{1}{2}\tan^{-1}\frac{2(6)}{9-4}=\frac{1}{2}\tan^{-1}\frac{12}{5}=\frac{1}{2}\tan^{-1}(2.4)=\frac{1}{2}(67^0)\approx 34^0$$

Exercise 8.4

1. Translate to parallel axes through the
   a. point (0,2), the equation $2x-y+2=0$.
   b. point (-1,2), the conic $x^2+y^2+2x-4y+1=0$.
   c. point (3,-4), the conic $x^2+2y^2-6x+16y+39=0$.
   d. point (-2,2), the conic $x^2+y^2-3xy+10x-10y+21=0$.
2. Transform to axes inclined at an angle
   a. 45^0 to the original axes of the conic $x^2-y^2=a^2$.
   b. 90^0 to the original axes of the conic $y^2=4px$.
   c. 45^0 to the original axes of the conic $x^2+y^2+4xy-1=0$.
   d. 45^0 to the original axes of the conic $x^2-y^2-2\sqrt{2}x-10\sqrt{2}y+2=0$.
3. Transform to new axes inclined at an angle
   a. $\tan^{-1}(1/2)$ to the original axes of the conic
   $14x^2+11y^2-36x+48y-4xy+41=0$ through (1,-2)
   b. $\tan^{-1}(-4/3)$ to the original axes of the conic
   $11x^2+4y^2-20x-40y+24xy-5=0$ through (2,-1)
   c. $\tan^{-1}(3/4)$ to the original axes of the conic
   $3x^2+10y^2+6x+52y-24xy=0$ through (3,1)

4. At what angle the axes are rotated about the origin so that the transformed equation of the conic

a. $11x^2+4y^2-20x-40y+24xy-5=0$ does not contain the term involving XY?

b. $5x^2+7y^2+2\sqrt{3}xy-16=0$. does not contain the term involving XY?

Glossary

Parabola:

- The eccentricity of the conic is e =c/a. The conic is

  ellipse , if e<1
  parabola, if e=1
  hyperbola, if e>1

- a. The standard form of the equation of a parabola that is symmetric with respect to the x–axis, with vertex V(0,0), focus F(p,0) and directrix the line x= –p is:

  $y^2=4px$

  b. The standard form of the equation of a parabola that is symmetric with respect to the y–axis, with vertex V(0,0), focus F(0,p) and directrix the line y= –p is:

  $x^2=4py$

- a. The standard form of the equation of a parabola that is symmetric with respect to the line y=k and with vertex V(h,k), focus F(h+p,k) and directrix line x=h–p is:

  $(y-k)^2=4p(x-h)$

  b. The standard form of the equation of a parabola that is symmetric with respect to the line x=h and with vertex V(h,k), focus F(h,k+p) and directrix line y=k–p is:

  $(x-h)^2=4p(y-k)$

- The equation of the tangent line at a point $P(x_1,y_1)$ to parabola is:

  $yy_1=2p(x+x_1)$

- The equation of any tangent to parabola $y^2 = 4px$ in the slope–form is:

$$y = mx + \frac{p}{m}$$

- The line y=mx+c should touch the parabola $y^2 = 4px$ under condition:

$$y = mx + \frac{p}{m},\ y^2 = 4py$$

- The equation of normal equation to parabola $y^2 = 4px$ at a point $P(x_1, y_1)$ is:

$$y - y_1 = \frac{-y_1}{2p}(x - x_1)$$

Ellipse:

- a. The standard form of the equation of an ellipse with center at the origin, length of the semi major axis **a**, length of the semiminor axis b and major axis along the x–axis is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

b. The standard form of the equation of an ellipse with center at the origin, length of the semimajor axis **a**, length of the semiminor axis **b** and major axis along the y–axis is:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

- a. The standard form of the equation of an ellipse with center at C(h,k), length of the semimajor axis **a** and semiminor axis **b**, and major axis parallel to the x–axis is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

b. The standard form of the equation of an ellipse with center at C(h,k), length of the semimajor axis **a** and semiminor axis **b**, and major axis parallel to the y–axis is:

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

- The equation of the tangent line to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ is:

$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1, \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \; at \; (x_1, y_1)$

- The equation of any tangent line to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the slope–form is:

$y=mx\pm\sqrt{a^2m^2+b^2}$

- The line y=mx+c should touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ under condition:

$c=\pm\sqrt{a^2m^2+b^2}$

- The normal equation to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ is:

$\frac{y-y_1}{y_1/b^2}=\frac{x-x_1}{x_1/a^2}$

Hyperbola:

- a. The standard form of the equation of a hyperbola with center at the origin and the x–axis as the transverse axis is:

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

b. The standard form of the equation of a hyperbola with center at the origin and the y–axis as the transverse axis is:

$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

- a. For horizontal standard form hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the asymptotes are the lines:

$y=\pm\frac{b}{a}x$

b. For vertical standard form hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, the asymptotes are the lines:

$y=\pm\frac{a}{b}x$

- a. The standard form of the equation of a hyperbola with center at C(h,k), vertices at $V_1(h+a,k)$ and $V_2(h-a,k)$, and foci at $F_1(h-c,k)$ and $F_2(h+c,k)$ is:

  $$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

  b. The standard form of the equation of a hyperbola with center at C(h,k), vertices at $V_1(h,k+a)$ and $V_2(h,k-a)$, and foci at $F_1(h,k+c)$ and $F_2(h,k-c)$ is:

  $$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

- a. For translating the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, horizontally, the asymptotes are the lines:

  $$(y-k)=\pm\frac{b}{a}(x-h)$$

  b. For translating the hyperbola $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$, vertically, the asymptotes are the lines:

  $$(y-k)=\pm\frac{a}{b}(x-h)$$

- The equation of the tangent line to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ is:

  $$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

- The equation of any tangent line to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in the slope–form is:

  $$y=mx\pm\sqrt{a^2m^2-b^2}$$

- The line y=mx+c should touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ under condition:

  $$c=\pm\sqrt{a^2m^2-b^2}$$

- The normal equation to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ is:

  $$\frac{-(x-x_1)}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$$

UNIT

9

DIFFERENTIAL EQUATIONS

This unit tells us, how to:

- define the differential equation, its order, degree general and particular solutions, and its identification as linear and nonlinear ordinary differential equations.
- demonstrate the concept in forming a differential equation.
- solve the first order linear and nonlinear ordinary differential equations by separable variable form, and homogeneous form and then how to reduce differential equations in the standard form of homogeneous.
- solve the real-life problems related to differential equations
- define the orthogonal trajectories and then how to show the orthogonal trajectories of the two families of curves

9.1 Introduction

The laws of the universe are written in the language of Mathematics. Algebra is sufficient to solve many static problems, but **the most interesting natural phenomena involve change and are describe only by equations that relates rates at which quantities change.**

Suppose the solution of problems concerning the motion of objects, the flow of charged particles, heat transport, etc often involves discussion of relations of the form

$$\frac{dx}{dt} = f(x,t) \quad \text{or} \quad \frac{dq}{dt} = g(q,t)$$

In the first equation, x might represent distance. For this case, $\frac{dx}{dt}$ is the rate of change of distance with respect to time t that is speed. In the second equation, q might be a charge and $\frac{dq}{dt}$ is the rate of flow of charge that is current. These are examples of **different equations**, so called because these are equations involving the derivatives of various quantities. Such equations arise out of situations in which change is occurring.

In engineering, differential equations are most commonly used to model dynamic systems. These are the systems which change with time. Examples include an electronic circuit with time-dependent currents and voltages, a chemical production line in which pressure, tank levels, flow rates, etc. vary with time.

There is a wide variety of differential equations which occur in engineering applications, and consequently there is a wide variety of solution techniques available.

i) *Definition of ordinary differential equation, order, degree, general and particular solutions*

Definition 9.1.1: [Differential Equation]: A differential equation is an equation that involves the derivatives of an unknown function (dependent variable) of one or more variables (independent variables).

If the unknown function depends on only one variable, then the derivative is an ordinary derivative, and the equation is then called an **ordinary differential equation.**

If the unknown function depends on more than one variable, then the derivative is partial derivative, and the equation is then called **partial differential equation.**

The following differential equations are the examples of ordinary differential equations with their corresponding unknown functions:

$$\frac{dy}{dx}=xy, \quad y(x)=? \qquad (1)$$

$$\frac{dy}{dx}=x+y, \quad y(x)=? \qquad (2)$$

$$\frac{dy}{dx}=\frac{x+y}{x-y}, \quad y(x)=? \qquad (3)$$

$$\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+y=3, \quad y(x)=? \qquad (4)$$

$$\left(\frac{d^3y}{dx^3}\right)^2-\frac{d^2y}{dx^2}+x^2\frac{dy}{dx}+y=3, \quad y(x)=? \qquad (5)$$

Definition 9.1.2: [Order of a Differential Equation]: The order of a differential equation is the order of the highest-order derivative occurring in the equation.

Definition 9.1.3:[Degree of a Differential Equation]: The degree of a differential equation is the power of the highest-order derivative occurring in the equation.

Example 9.1.1: Determine the order and degree of the following ordinary differential equations:

a. $\frac{dy}{dx}=\frac{x+y}{x-y}$

b. $\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+y=3$

c. $\left(\frac{d^3y}{dx^3}\right)^2 - \frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 3$

Solution: Differential equation (a) is an ordinary differential equation of order 1 and degree 1, since the highest ordinary derivative is of order 1 and the exponent of the highest ordinary derivative is 1. Differential equation (b) is an ordinary differential equation of order 2 and degree 1, while Differential equation (c) is an ordinary differential equation of order 3 and degree 2.

Solution of a Differential Equation: A solution of an equation in a single variable is **a number** which satisfies the equation. In similar fashion, solutions of the differential equations are **functions,** rather than numbers, which satisfy the differential equation. The variables which appear in equations are called **"unknowns."** Exactly, the only dependent variable in differential equations is referred to as **"unknown."**

For illustration, a solution of the differential equation dy/dx=1 is an expression of the unknown dependent variable y in terms of the independent variable x.

Definition: 9.1.4:[Solution of a Differential Equation]:A solution of an ordinary differential equation is any function $y = f(x)$ or $f(x, y)$ which when substituted in the differential equation, reduces the differential equation to an identity' that is, it satisfies the equation.

Example 9.1.2: Show that $y = x + A$ is a solution of the first order differential equation $dy/dx = 1$.

Solution: The given function y=x+A and its derivative dy/dx=1 is used in the differential equation dy/dx=1 to obtain:

$$\frac{dy}{dx} = 1$$

$$1 = 1, \textit{ identity left side} = \textit{right side}$$

This shows that $y = x + A$ is a solution of the ordinary differential equation $dy/dx = 1$.

The solution that depends on an arbitrary constant quantity is called the **general solution** of a differential equation. For example, by choosing different values of an arbitrary constant quantity, different solutions of a differential equation are

obtained. This means that the general solution represents a family of curves (many solution curves) for any choice of **arbitrary constant quantity.**

Thus, the solution $y = x + A$ of a differential equation $dy/dx = 1$ is declared the general solution. The general solution $y = x + A$ represents a family of curves (parallel lines) for any choice of arbitrary constant A.

If we give definite value to arbitrary constant quantity in the general solution, the solution so obtained is called **a particular solution.** For example, to determine a particular value of A for particular solution (line), we need to be given more information in the form of **initial condition.** For example, if we are given x=0, y=1, then from $y = x + A$, we have the definite value of arbitrary constant A:

$$y = x + A \quad \Rightarrow \quad 1 = 0 + A \quad \Rightarrow \quad A = 1$$

The definite value of A is used in the general solution $y = x + A$ to obtain the **particular solution** (line) $y = x + A = x + 1$ which additionally satisfies the initial condition y(0)=1.

Definition 9.1.5: [General and Particular Solution]: The solution of a differential equation when depends on a single arbitrary constant quantity, is then called the **general solution** of the first order different equation. If we give particular steps for value to a single arbitrary constant quantity, then the solution to obtain is called the **particular solution** or **specific solution** or **exact solution** or **actual solution** of first order differential equation.

Graphically,

- the general solution of a first order deferential equation represents a family of curves for any choice of arbitrary constant quantity.
- The particular solution of a first order differential equation is a particular curve chosen from a family of curves (general solution) for a particular value of a constant quantity.

Example 9.1.3:[General & Particular Solution]: Graphically, show that $y = x + A$ is a general solution of the first order differential equation $dy/dx = 1$. Find a particular solution, when $x = 0$ and $y = 1$.

Solution: The general solution $y = x + A$ of a first order differential equation $dy/dx = 1$, represents a family of parallel straight lines for different values of arbitrary constant quantities $A = 0, 1. 2, \ldots$

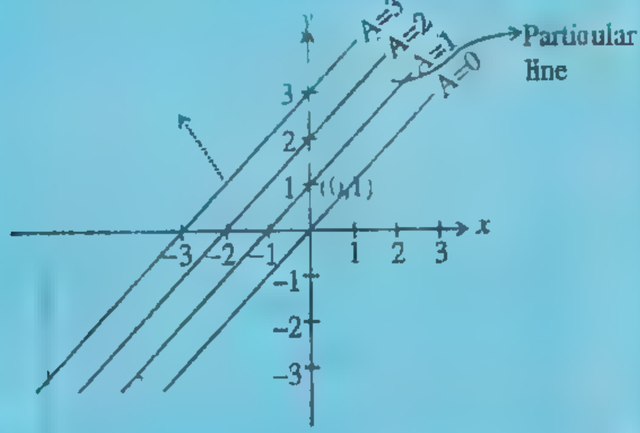


Fig. 9.1

The particular value for the particular line that passes through a point P(0, 1) can be found from the general solution y=x+A by putting x=0, y=1:

$$y = x + A \Rightarrow 1 = 0 + A \Rightarrow A = 1$$

Use this particular value of A=1 in general solution y=x+A to obtain a particular solution (line) $y = x + 1$.

If we are to determine the solutions of a differential equation subject to conditions on the unknown function and its derivatives specified for one value of the independent variable, the conditions are then called **initial conditions** and the related differential equation is called an **initial value problem** "IVP".

Thus, the problem of example 10.1.3 is the initial value problem that leads the notation:

$$\frac{dy}{dx} = 1, \quad y(0) = 1, \quad IVP \qquad (6)$$

Example 9.1.4: Determine a particular solution for the first order differential equation $ds/dt = -32\,ft/\sec$ that satisfies the initial condition s=0, when t=0.

Solution: This information develops the initial value problem

$$ds/dt = -32, \; s(0) = 0$$

for which the solution is the unknown function s(t) that can be found by integrating directly the first order differential equation with respect to t:

$$\frac{ds}{dt} = -32$$

$$\int \frac{ds}{dt} = \int -32dt + A$$

$$s(t) = -32t + A$$

The general solution $s(t) = -32t - A$ at a point P(0, 0) is giving $A = 0$. Use this $A = 0$ in general solution to obtain the particular solution $s(t) = -32t$.

Another classification of an ordinary differential equation is to determine whether it is linear or non-linear.

Definition 9.1.6: [Linear Differential Equation]: A differential equation is said to be **linear**, if the dependent variable and its derivatives occur to the **first power** only and if there are no products involving the dependent variable and/or its derivatives. There should be no non-linear functions of the dependent variable, such as Sine, Exponential, etc. A differential equation which is not linear is said to be non-linear. The linearity of a differential equation is not affected by the presence of non-linear terms involving the independent variable.

The n^{th} order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_o(x)y = f(x), \qquad (7)$$

is linear in y (dependent variable) and its all derivatives $dy/dx, d^2y/dx^2, \ldots, d^n y/dx^n$. Here $f(x)$, $a_n(x), \ldots, a_1(x)$, $a_o(x)$ are functions of x (or real constant quantities) and $a_n(x)$ is not zero.

Example 9.1.5:[Linear & Non-linear]: The ordinary differential equations are

$$\frac{dv}{dt} = -32, \qquad \frac{d^2 s}{dt^2} = -32, \qquad \frac{dy}{dx} = x + 1, \qquad \frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} + y = 3$$

are linear differential equations, while the differential equation

$$\frac{d^2 y}{dx^2} + 4y\left(\frac{dy}{dx}\right)^2 + 2y = \cos x$$

is non-linear differential equation, since there is a non-linear term which is the product of y and its derivatives dy/dx whose exponent is 2.

Sometimes the given differential equation, say,

$$y\frac{dy}{dx} = y^2x + xy$$

is not linear, but can be reduced into linear form (7):

$$y\frac{dy}{dx} = y^2x + xy, \quad \text{non-linear}$$

$$\frac{dy}{dx} = \frac{y^2x}{y} + \frac{xy}{y}, \quad \text{divide out by } y$$

$$\frac{dy}{dx} = xy + x$$

$$\frac{dy}{dx} - xy = x, \quad \text{linear}$$

The differential equations

$$\frac{dy}{dx} = \frac{x+y}{x-y}, \qquad \left(\frac{d^3y}{dx^3}\right)^2 - \frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 3$$

are non-linear ordinary differential equations. It is not possible to reduce them into linear form (7).

9.2 Formation of Differential Equation

The steps for forming the differential equations are the following:

1. Discover the differential equation that describes a specified physical situation.
2. Find either exactly or approximately, the appropriate solution of that equation.
3. Interpret the solution that is found.

i) The concept of formation of a differential equation

The mathematical model of a real-life problem likes,

1. the rate at which the distance travels by you is 30 mph (have you seen on the right hand side of the roads). Find the total distance travelled by you at a time t hours.

   The rate at which the distance travelled, is modeled by a first order differential equation:

   $$\frac{dS}{dt} = 30, \qquad \text{the per hour speed}$$

   Here $S(t)$ is the unknown distance travelled by you w.r.t t number of hours, and the rate at which the distance travelled is the first derivative of $S(t)$ with respect to t.

Integrating with respect to t to obtain S(t)

$$\int \frac{dS}{dt} = 30$$
$$S(t) = 30t + A$$

the distance traveled by you with respect to t number of hours and the constant quantity A is the fixed distance in this situation.

2. The rate at which the animal population is growing at a constant rate 4%. The habitat will support no more than 10,000 animals. There are 3000 animals present now. Find an equation that gives the animal population y w.r.t x number of years.

The rate at which the animal population grows, is modeled by a first order differential equation:

$$\frac{dP}{dx} \propto (N - P)$$
$$\frac{dP}{dx} = k(N - P),\ k \text{ is the constant of proportionality}$$

Here P(x) is the unknown animal population w.r.t x number of years and the rate at which the animal population grows, is the first derivative of P(x) with respect to x:

$$\frac{dP}{dx} = k(N - P)$$
$$= 0.04(10,000 - P)$$
$$\frac{dP}{(10,000 - P)} = 0.04dx$$

Here k =0.04 is the constant growth, N=10,000, is the total size of the animal population in that habitat.

Integrating with respect to x to obtain P(x)

$$\int \frac{dP}{(10,000 - P)} = \int (0.04)dx + A$$

the total population and A, the fixed population that depends on P=3000 when x=0. This problem is the IVP problem with the initial condition P(0)=3000.

Exercise 9.1

1. Find the order and degree of each the following ordinary differential equations:

a. $\frac{dy}{dx} = x^2 + y$

b. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 11y = 3x$

c. $\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)^3-y=0$

Also indicate the linear and non-linear in the above differential equations.

2. In each case, show that the indicated function is a solution of the differential equation:

a. $y=e^x+e^{2x}$, $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$

b. $y=x-x\ln x$, $x\frac{dy}{dx}+x-y=0$

c. $y=(x+c)e^{-x}$, $\frac{dy}{dx}+y=e^{-x}$

3. For each of the following equations, determine whether or not it becomes linear when divided by dx or dy:

a. $(x+y)dy=(x-y)dx$ b. $ady+by\sin x dx=0$

c. $3ydx+2xdy=0$ d. $e^x dy+xy^{\frac{1}{3}}dx=0$

4. In each case, find the particular solution, when the initial condition and the general solution of the differential equation are the following:

a. $xy=c$, $y(2)=1$ b. $y=x-x\ln x+c$, $y(1)=2$

c. $\sin(xy)+y=c$, $y(\pi/4)=1$ d. $\frac{y^2}{x}=\frac{x^2}{2}+c$, $y(1)=1$

5. Solve the following initial value problems:

a. $\frac{dy}{dx}=\cos x$, $y(0)=1$ b. $\frac{dy}{dx}=x^2$, $y(0)=1$

c. $\frac{dy}{dx}=2xy^2$, $y(3)=-1$ d. $\frac{dy}{dx}+y=y^2$, $y(0)=1/2$

e. $y\frac{dy}{dx}+xy^2-x=0$, $y(0)=-1$ f. $2\frac{dy}{dx}=4xe^{-x}$, $y(0)=42$

9.3 Solving differential equations

If the solution of a first order differential equation is not possible by direct integration, then, the integral process (in case of difficulties) for obtaining the

solution of a differential equation indicates the actual concept of a differential equation.

i) Solutions of first order and first degree differential equations

We examine techniques for solving first order differential equations. For this unit, the recommended techniques for solving the differential equations are the separation of variables, reducible to separation form, homogeneous and equations reducible to homogeneous form.

- ***Separable differential equation***

If the solution of a differential equation is not possible by direct integration, then the integral technique called **separation of variables** will be used for solving the differential equation. Separation of variables is a technique commonly used to solve first order differential equations. It is so called because we try to rearrange the equation to be solved in such a way that all terms involving the dependent variable (y say) appear on one side of the equation, and all terms involving the independent variable (x, say) appear on the other side. It is not possible to rearrange all first order differential equations in this way so this technique is not always appropriate. Further, it is not always possible to perform the integration even if the variables are separable.

In general, a differential equation of the form

$$\frac{dy}{dx}=\frac{f(x)}{g(y)},\ g(y)\neq 0 \qquad (8)$$

that by shifting x on one side and y on the other side

$$g(y)dy=f(x)dx \qquad (9)$$

is giving a **separable differential equation**. The solution to separable differential equation (9) can be found by integrating left hand side w.r.t y and right hand side w.r.t x.

Example 9.3.1:[General Solution]: Find the general solution of the linear differential equation $\frac{dy}{dx}=y$.

Solution: The solution of the given linear differential equation is not possible by direct integration. The separable form of the given first order differential equation is obtained by shifting y on the left and x on the right:

$$\frac{1}{y}dy = dx$$

that by integration w.r.t x

$$\int \frac{1}{y}dy = \int dx$$

$$\ln y = x + c$$

$$y = e^{x+c}$$

$$= e^x e^c = c_1 e^x, \quad c_1 = e^c \qquad (10)$$

is giving the general solution of the first order differential equation. This general solution represents a family of exponential functions as shown in the figure (9.2).

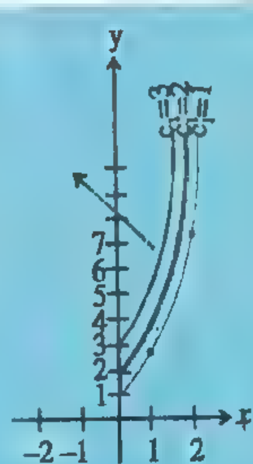


Fig. 9.2

- ***Differential equations reducible to separable form***

If the solution of the differential equation is not possible by separable form, then the given differential equation can be reduced in separable form by substitution. This substitution changes the dependent variable form y to a new variable, say, u and keeps x as the independent variable.

Example 9.3.2:[Reducible to Separable Form]: Find the general solution of the non-linear differential equation $\dfrac{dy}{dx} = (x+y)^2$.

Solution: The given non-linear differential equation is not separable differential equation, but can be reduced into separable form by substitution,

$$x+y=u(x)$$

that on differentiation w.r.t x is giving:

$$\frac{d}{dx}(x+y)=\frac{du}{dx} \Rightarrow 1+\frac{dy}{dx}=\frac{du}{dx} \Rightarrow \frac{dy}{dx}=\frac{du}{dx}-1$$

Use x+y=u and dy/dx=(du/dx)-1 in the given differential equation to obtain separable differential equation in variable u and its derivative du/dx:

$$\frac{du}{dx}-1=u^2$$

$$\frac{du}{dx}=1+u^2$$

$$\frac{du}{1+u^2}=dx \qquad (11)$$

Integrating equation (11) to obtain the general solution of ordinary differential equation (11)

$$\int\frac{du}{1+u^2}=\int dx$$

$$\tan^{-1}u=x+c$$

$$u=\tan(x+c)$$

that by back substitution of $u=x+y$ is giving

$$x+y=\tan(x+c)$$

$$y=-x+\tan(x+c)$$

the general solution of the given ordinary differential equation that depends on a single arbitrary constant c.

If the differential equation is not reducible to separable form by substitution, then the differential equation might be a homogeneous differential equation. This can be solved by the procedure developed is as under:

- ***Homogeneous differential equation***

The homogeneous differential equations are related to homogeneous functions that are already discussed in detail in Unit-11.

Definition 9.3.1:[Homogeneous Function]: A function f(x,y) is homogeneous function of degree n in variables x and y if and only if for all values of the variables x, and y and for every positive value of λ, the identity is true.

$$f(\lambda x,\lambda y)=\lambda^{n} f(x, y), \quad n=1,2,3,\ldots \qquad (12)$$

For illustration, the function $f(x,y)=x^2+y^2$ is homogeneous function of degree 2, since the identity (12) is true:

$$f(\lambda x,\lambda y)=(\lambda x)^2+(\lambda y)^2=\lambda^2\left(x^2+y^2\right)=\lambda^2 f(x,y),\ x=\lambda x, y=\lambda y$$

The identity (12) is not true for a function $f(x,y)=x^2+y^2+1$, since the function is not homogeneous.

Definition 9.3.2:[Homogeneous Differential Equation]: The differential equation

$$\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)} \qquad (13)$$

is called a homogenous differential equation, if it defines a homogenous function of degree zero.

The homogeneous differential equation (13) can be reduced to separable form by introducing a new variable:

$$u(x)=\frac{y}{x} \ \text{or}\ y=ux \quad \text{and} \quad \frac{dy}{dx}=\frac{d}{dx}(ux)=u+x\frac{du}{dx} \qquad (14)$$

The substitution of (14) in equation (13) automatically converts the homogeneous differential equation in separable differential equation.

Example 9.3.3:[Homogeneous Differential Equation]: Find the general solution of the homogeneous differential equation:

$$\frac{dy}{dx}=\frac{y-x}{y+x}$$

Solution: The given differential equation defines a homogeneous function of degree zero, when the function on the right of the given differential equation defines a homogeneous function of degree zero:

$$\frac{dy}{dx}=\frac{y-x}{y+x}$$

$$=\frac{\lambda y-\lambda x}{\lambda y+\lambda x}$$

$$=\frac{\lambda(y-x)}{\lambda(y+x)}$$
$$=\frac{\lambda}{\lambda}\left[\frac{y-x}{y+x}\right]$$
$$=\lambda^{1-1}\left[\frac{y-x}{y+x}\right]=\lambda^{0}\left[\frac{y-x}{y+x}\right]=\left[\frac{y-x}{y+x}\right],\ HDE$$

The given homogeneous differential equation is used for the assumptions

$y=ux,\ \frac{dy}{dx}=u+x\frac{du}{dx}$ to obtain a separable differential equation of the form:

$$u+x\frac{du}{dx}=\frac{ux-x}{ux+x}=\frac{u-1}{u+1}$$
$$x\frac{du}{dx}=\frac{u-1}{u+1}-u$$
$$=\frac{u-1-u^{2}-u}{u+1}=\frac{-(u^{2}+1)}{u+1}$$
$$-\frac{(u+1)}{u^{2}+1}du=\frac{dx}{x},\ SDE \qquad (15)$$

Integrating SDE (15) to obtain the general solution of the SDE (15):

$$-\int\frac{(u+1)du}{u^{2}+1}=\int\frac{dx}{x}$$
$$-\int\frac{2(u+1)du}{2(u^{2}+1)}=\int\frac{dx}{x},\quad \textit{Multiply and divide out by 2}$$
$$-\frac{1}{2}\int\frac{2udu}{u^{2}+1}-\int\frac{du}{u^{2}+1}=\ln x+c,$$
$$-\frac{1}{2}\ln\left(u^{2}+1\right)-\tan^{-1}u=\ln x+\ln c$$
$$-\ln\sqrt{u^{2}+1}-\tan^{-1}u=\ln cx$$
$$-\tan^{-1}u=\ln\sqrt{u^{2}+1}+\ln cx$$
$$=\ln cx\sqrt{u^{2}+1}$$
$$\tan^{-1}u=-\ln cx\sqrt{u^{2}+1} \qquad (16)$$

The back substitution $u = y/x$ is used in equation (16) to obtain the general solution of the given homogeneous differential equation:

$$\tan^{-1}\frac{y}{x} = -\ln cx\sqrt{\frac{y^2}{x^2}+1}$$

$$\tan^{-1}\frac{y}{x} = -\ln c\sqrt{y^2+x^2} \Rightarrow \frac{y}{x} = \tan\left[-\ln c\sqrt{y^2+x^2}\right]$$

- ***Differential equations reducible to homogeneous differential equations***

Example 9.3.4:[Reducible to Homogeneous]: Find the general solution of the differential equation $x\frac{dy}{dx} = x + y$.

Solution: The given differential equation is not in the standard form of homogeneous differential equation, but it can be reduced in the standard form of homogenous differential equation by the following procedure:

Divide out by x to obtain the standard form of homogeneous differential equation:

$$\frac{dy}{dx} = \frac{x+y}{x}, \ HDE \qquad (17)$$

Homogeneous differential equation (17) is used for the assumptions

$y = ux$, $dy/dx = u + xdu/dx$ to obtain a separable differential equation of the form:

$$u + x\frac{du}{dx} = \frac{x+xu}{x}$$

$$= 1 + u$$

$$x\frac{du}{dx} = 1 \Rightarrow du = \frac{dx}{x}, \ SDE \qquad (18)$$

Integrating the SDE (18) to obtain the general solution of the SDE (18):

$$\int du = \int \frac{dx}{x}$$

$$u = \ln x + \ln c = \ln cx$$

that by back substitution $u = \frac{y}{x}$ is giving

$$\frac{y}{x} = \ln cx$$
$$y = x \ln cx$$

the general solution of the given homogeneous differential equation that depends on a single arbitrary constant c.

If the differential equation is not homogeneous differential equation, then it might be a nonhomogeneous differential equation. The nonhomogeneous differential equation is beyond of this curriculum.

ii) Real-life problems related to differential equations

Natural Growth and Decay: The differential equation

$$\frac{dy}{dt} \alpha\, y \Rightarrow \frac{dy}{dt} = k\,y, \quad k \text{ is constant of proportionality} \qquad (19)$$

is a mathematical model, serves as remarkable wide range of natural phenomena. Any quantity involving whose time rate of change is proportional to its current value. For example, in problems to determine rates of decay, heating and cooling, compound interest, evaporation, mixture, rumor and others. A positive value of k indicates growth, while a negative value of k indicates decay.

In problems, such as space restrictions (a limited amount of food or maximum size of population) tend to inhibit growth of populations as time goes on, the rate of growth of population is proportional to how close the population is to that maximum size N:

$$\frac{dy}{dt} = k(N - y) \qquad (20)$$

The constant k is called the growth rate constant, while N represents the maximum size of the population.

Example 9.3.5:[Real-Life Problem; IVP]:A certain bacteria grows at a rate that is proportional to the number present at a particular time. If the number of bacterial at a time t=0 is N_o and at time $t=1$ hour, the number of bacteria is $5N_o/2$. Determine the time necessary for the number of bacteria to be quadruple.

Solution: If N(t) is the unknown number of bacteria w.r.t time t hours, then, the rate at which bacterial grows, is represented by:

$$\frac{dN}{dt} \alpha N \Rightarrow \frac{dN}{dt} = kN \tag{21}$$

Reduce the differential equation to separable form

$$\frac{dN}{N} = k\,dt$$

that on integration is giving the general solution of (21):

$$\int \frac{dN}{N} = \int k\ dt$$

$$\ln N = kt + c \tag{22}$$

The initial condition $N(0) = N_o$ is used in equation (22) to obtain c:

$$\ln N_o = 0 + c \Rightarrow c = \ln N_o \tag{23}$$

Use c in equation (22) to obtain a particular solution:

$$\ln N = kt + \ln N_o \Rightarrow \ln \frac{N}{N_o} = kt \Rightarrow \frac{N}{N_o} = e^{kt} \Rightarrow N = N_o e^{kt} \tag{24}$$

The condition $N(1) = 5\,N_o/2$ is used in equation (24) to obtain the value of k:

$$5\,N_o/2 = N_o e^k \Rightarrow e^k = 5/2 \Rightarrow k = \ln(5/2) = 0.9163 \tag{25}$$

Use the value of k in equation (24) to obtain a particular solution (specific number of bacteria):

$$N = N_o e^{0.9163t} \tag{26}$$

The condition $N = 4N_o$ (when the bacterial have quadrupled) is used in equation (26) to obtain the time

$$4N_o = N_o e^{0.9163t}$$

$$4 = e^{0.9163t} \Rightarrow 0.9163t = \ln 4 \Rightarrow t = \ln 4/0.9163 = 1.51\ hr$$

at which the bacteria is four times of the original number of bacteria.

Example 9.3.6:[Maximum Production]: The rate at which a new worker in a certain factory produces items, is given by

$$\frac{dy}{dx} = k(N-y) = 0.2(125-y),\ \ k = 0.2,\ N = 125$$

Where y is the number of items produced by the worker per day, x is the number of days worked and the maximum production per day is 125 items. Assume that the

worker produced 20 items the first day on the job x = 0. How many items will be produced by the worker on the job of x = 10 (days)?

Solution: If y(x) is the unknown number of items produced by the worker w.r.t x number of days, then the rate at which the items produced, was approximated by:

$$\frac{dy}{dx}=k(N-y)=0.2(125-y), \quad k=0.2,\ N=125$$

Reduce the given differential equation to separable form

$$\frac{dy}{(125-y)}=0.2\,dx$$

that on integration is giving a general solution (general items production):

$$\int\frac{dy}{125-y}=\int 0.2\,dx$$

$$\frac{\ln(125-y)}{-1}=0.2x+c$$

$$\ln(125-y)=-0.2x-c$$

$$(125-y)=e^{-0.2x-c}$$

$$=e^{-0.2x}e^{-c}$$

$$=c_1e^{-0.2x}, \qquad c_1=e^{-c}$$

$$y=125-c_1e^{-0.2x} \qquad (27)$$

The initial condition y = 20 for x = 0 is used in equation (27) to obtain the particular value of constant c:

$$y=125-c_1e^{-0.2x}$$

$$20=125-c_1e^{0} \Rightarrow c_1=125-20=105$$

$c_1=105$ is used in equation (27) to obtain a particular solution y(x) (the specific number of items produced by the worker in x number of days):

$$y(x)=125-105e^{-0.2x} \qquad (28)$$

The items produced by the worker on the job of x = 10 days is obtained by putting x = 10 in equation (28):

$$y(10)=125-105e^{-2}=110.825 \text{ items}$$

9.4 Orthogonal Trajectories

Our experience with first order differential equations has taught us that such equations often have general solutions containing a single arbitrary constant. Each such solution defines a corresponding set of integral curves. A nonempty set of plane curves defined by a differential equation involving just one parameter (single arbitrary constant) is commonly called a one-parameter family of curves. Of special importance in certain applications are those one-parameter families of curves which are orthogonal trajectories of one another.

Definition 9.4.1:[Orthogonal Trajectories]: The curves of a family $F(x,y,c_1)$ are said to be orthogonal trajectories of curves of a family $G(x,y,c_2)$, if and only if each curve of either family is intersected by at least one curve of the other family and at every point of intersection of a curve of F with a curve of G, the two curves are perpendicular.

i) Orthogonal trajectories of the given family of curves

The two families of curves $F(x,y,c_1)$ and $G(x,y,c_2)$ are perpendicular at a point of intersection, if and only if their tangents are perpendicular at the point of intersection. If their tangent lines, say, L_1 and L_2, are perpendicular, then the product of their slopes equals -1:

$m_1 m_2 = -1$, *m_1 and m_2 are the slopes of the two tangent lines L_1 and L_2*

$$m_1 = -\frac{1}{m_2} \Rightarrow \left(\frac{dy}{dx}\right)_G = -\frac{1}{\left(\frac{dy}{dx}\right)_F} \qquad (29)$$

This is called the **differential equation of orthogonal trajectories**. If one family of curves F is given, then the other family of curves G can be found by solving the differential equation of orthogonal trajectories (29).

Example 9.4.1:[Orthogonal Trajectories]: Determine the orthogonal trajectories of the family of curves (circles) $x^2 + y^2 = c$.

Solution: To determine the orthogonal trajectories of the circles, we need to determine the slope (derivative) of the family of circles

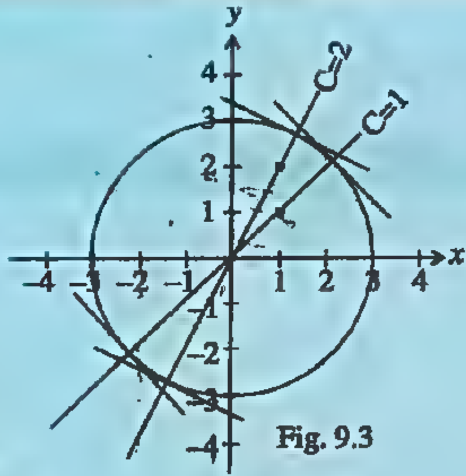


Fig. 9.3

$x^2 + y^2 = c$

with respect to x

$$2x + 2y\frac{dy}{dx} = 0$$

$$\left(\frac{dy}{dx}\right)_F = -\frac{x}{y} \qquad (30)$$

The differential equation of the orthogonal trajectories (29) with the slope of the given orthogonal trajectories (30) is used to obtain the other family of curves G of orthogonal trajectories:

$$\left(\frac{dy}{dx}\right)_G = \frac{-1}{\left(\frac{dy}{dx}\right)_F}$$

$$\frac{dy}{dx} = \frac{-1}{-\frac{x}{y}} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}, \ SDE$$

$$\int\frac{dy}{y} = \int\frac{dx}{x}$$

$$\ln y = \ln x + \ln C$$

$$y = Cx \qquad (31)$$

Thus, the family of curves G represents a family of homogeneous straight lines that pass through the origin. This is the result, we would expect, since the radii of a circle are the homogeneous lines (y=Cx, C is any real number) perpendicular to the lines tangent to a circle.

Exercise 9.2

1. Find general solution of the following differential equations

a. $\dfrac{2x\dfrac{dy}{dx}-2y}{x^2}=0$ b. $\dfrac{dy}{x}+ydx=2dx$

c. $\left(\dfrac{dy}{dx}\right)^2=1-y^2$ d. $e^x\dfrac{dy}{dx}+y^2=0$

e. $\sqrt{1-x^2}\,dy=\sqrt{1-y^2}\,dx$ f. $\operatorname{cosec}^2 x\,dy+\sec y\,dx=0$

2. Reduce the following differential equations in separable form and then solve:

a. $y'=(y+x)^2$ b. $y'=\tan(x+y)-1$

c. $y'=(x+e^y-1)e^{-y}$ d. $y\dfrac{dy}{dx}+xy^2-x=0$

e. $\dfrac{dy}{dx}=\dfrac{y}{x}+\dfrac{x^2}{y^2}$ f. $(y-3)dy=(x^3+1)dx$

3. Solve the following homogeneous differential equations:

a. $\dfrac{dy}{dx}=\dfrac{x+y}{x-y}$ b. $\dfrac{dy}{dx}=\dfrac{xy-y^2}{x^2}$ c. $\dfrac{dy}{dx}=\dfrac{x^2+3y^2}{2xy}$

d. $\dfrac{dy}{dx}=\dfrac{\sqrt{x^2-y^2}+y}{x}$ e. $\dfrac{dy}{dx}=\dfrac{xy+y^2}{x^2+xy+y^2}$

4. Reduce the following differential equations in the standard form of homogeneous form and then solve:

a. $x\dfrac{dy}{dx}=y+\sqrt{x^2+y^2},\ y(4)=3$ b. $(x^4+y^4)dx=2x^3y\,dy,\ y(1)=0$

5. The slope of a family of curves at a point P(x,y) is $\dfrac{y-1}{1-x}$. Determine the equation of the curve that passes through the point P(4, –3).

6. Find the solution curve of the differential equation $xyy' = 3y^2 + x^2$ which passes through the point P(-1, 2).

7. Determine the particular solution $y = f(t)$ of the homogeneous differential equation $t^2 y' = y^2 + 2ty$ with initial condition $y(1) = 2$.

8. A particle moves along the x-axis so that its velocity at any point is equal to half its abscissa minus three times the time. At a time $t = 2$, $x = -4$, determine the motion of a particle along the x-axis.

9. The rate of consumption of oil (billions of barrels) is given by $\frac{dx}{dt} = 1.2e^{0.04t}$. Where $t = 0$ correspond to 1990. Find the total amount of oil used from 1990 to year 1995. At this rate, how much oil will be used in $8(t = 8)$ years?

10. The rate of infection of a disease (in people per month) is given by:

$$\frac{dI}{dt} = \frac{100t}{t^2 + 1}$$

Where t is the time in months since the disease broke out. Find the total number of infected people over the first four months of the disease.

11. The rate of reaction to a drug is given by

$$\frac{dR}{dx} = 2x^2 e^{-x}$$

Where x is the number of hours since the drug was administered. Find the total reaction to the drug form $x = 1$ to $x = 6$.

12. The rate of increase of the number of cellular phone subscribers (in millions) since services began, was given by:

$$\frac{ds}{dx} = 0.38x + 0.04$$

Where x is the number of years since 1998, when the services started. There were 0.25 million subscribers in year 1998 (x=0). Find a function that gives the number of subscribers for the year 2004.

13. Determine the equations of the orthogonal trajectories of the following families of curves:

a. $y = cx^3$	b. $xy = c$	c. $y = cxe^x$
d. $y^2 = x^2 + c$	e. $y = c\sin 2x$	f. $e^x \cos y = c$
g. $y = \sqrt{x + c}$	h. $y = x^2 + c$	i. $e^x \sin y = c$
j. $\cos x \cosh y = c$	k. $e^x(x\cos y - y\sin y) = c$	

GLOSSARY

Glossary

- **Differential Equation:** A differential equation is an equation that involves the derivatives of an unknown function (dependent variable) of one or more variables (independent variables).
- **Order of a Differential Equation:** The order of a differential equation is the order of the highest-order derivative which appears in the equation.
- **Degree of a Differential Equation:** The degree of a differential equation is the power of the highest-order derivative which appears in the equation.
- **Solution of a Differential Equation:** A solution of an ordinary differential in one dependent variable y on an interval I is a function $y(x)$ which, when substituted for the dependent variable y and its derivatives $y', y'', \ldots$, reduces the differential equation to an identity in the independent variable x over interval I.
- **Linear Differential Equation:** A differential equation is **linear** in a set of one or more of its dependent variables if and only if each term of the equation which contains a variable of the set or any of their derivatives is of the first degree in those variables and their derivatives.
- **Homogeneous Function:** A function f(x,y) is homogeneous function of degree n in variables x and y if and only if for all values of the variables x, and y and for every positive value of λ, the identity is true:

  $f(\lambda x, \lambda y) = \lambda^n f(x, y), \quad n = 1, 2, 3, \ldots$
- **Homogeneous Differential Equation:** The differential equation

  $$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

  is called a homogenous differential equation, if it defines a homogenous function of degree zero.

UNIT

10 PARTIAL DIFFERENTIATION

This unit tells us, how to:

- define a function of two variables.
- define partial derivative of a function of two variables, partial derivatives of a function w.r.t x and y variables and its linkage to real-life problems.
- define homogeneous function of degree n, and to prove Euler's theorem based on homogeneous functions.
- verify the Euler's theorem for homogeneous functions of different degrees.

10.1 Differentiation of Function of Two variables

The goal of this unit is to extend the methods of single variable differential calculus to functions of two variables. In many practical situations, the value of one quantity may depend on the values of two or more others. For example, the amount of water in a reservoir may depend on the amount of rainfall and on the amount of water consumed by local residents. The current in an electrical circuit may vary with the electromotive force, the capacitance, the resistance, and the impedance in the circuit. The flow of blood from an artery into a small capillary may depend on the diameter of the capillary and the pressure in both the artery and the capillary. The output of a factory may depend on the amount of capital invested in the plant and on the size of the labor force. We will analyze such situations using functions of several variables.

In many problems involving functions of several variables, the goal is to find the derivative of the function with respect to one of its variables when all the others are held constant. In this unit, we need to develop the concept and shall see how it can be used to find slopes and rates of change in case of two variables function.

i) *Definition of a function of two variables*

In the real world, physical quantities often depend on two or more variables. For example, we might be concerned with the temperature on a metal plate at various points at time t. The locations of temperature on the plate are given as ordered pairs (x,y), so that the temperature T can be considered as a function of two location variables x and y, as well as a time variable t. The notation of a function of single variable, we might extend this as T(x,y,t). We begin our study of function of two variables by examining this notation and a few other basic concepts.

For illustration, if a company produces x items at a cost of 10 rupees per item, then the total cost C(x) of producing x items is given by:

$C(x) = 10x$

The cost is a function of one independent variable, the number of items produced. If the company wants to produce two products, with x of one product at a cost of rupees 10 each, and y of another product at a cost of rupees 15 each, then the total cost to the firm is a function of two independent variables (two inputs) x and y:

$C(x, y) = 10x + 15y$

When $x=5$ and $y=12$, the total cost is written with $C(5,12) = 10(5) + 15(12) = 230$ rupees. This is the linkage of a function of two variables in real-life situation problems.

Definition 10.1.1: [Function of Two Variables]: A function $z = f(x, y)$ is a **function of two variables x and y**, if a unique of z is obtained from each ordered pair of real numbers (x, y). The real numbers x and y are **independent variables;** z is the **dependent variable.** The set of all ordered pairs of real numbers (x, y) such that $f(x, y)$ is a real number, is **the domain** of f and the set of all values of $f(x, y)$ is **the range.**

Example 10.1.1:[Function of two Variables]: The area $A(r) = \pi r^2$ of a circle depends on the radius r is a single variable function. But the volume $V(r, h) = \pi r^2 h$ of a cylinder (depends on the radius r and height h), constitutes a function of two independent variables r and h.

Example 10.1.2:[Domain and Range]: How to show that $z = f(x, y) = \sqrt{1 - x + y}$ is a function of two independent variables x and y? Find also the domain and range of a function of a given function.

Solution: For this function, we need to show the transformation of two independent variables (inputs) x and y in just a single dependent variable (output) z. In respect of any two real values of independent variables x and y, say, $x = 2$ and $y = 1$, the function $z = f(2,1) = \sqrt{1 - 2 + 1} = 0$ gives response of just one real value of z which is $z = 0$.

The function $z = f(x, y) = \sqrt{1 - x + y}$ is therefore declared a function of two independent variables x and y.

The domain of f (x, y) is the set of all ordered (x, y) for which $\sqrt{1 - x + y}$ is defined. We must have $1 - x + y \geq 0$ or $y \geq x - 1$, in order for the square root to be defined.

In a function $z = f(x, y) = \sqrt{1 - x + y}$, we see that $z = f(x, y)$ must be nonnegative and the range of $f(x, y)$ is all $z \geq 0$ (because square roots are nonnegative).

Example 10.1.3:[Function of Two variables]: Let x represent the number of milliliters (ml) of carbon dioxide released by the lungs in 1 minute. Let y be the change in the carbon dioxide content of the blood as it leaves the lungs (y is measured in ml of carbon dioxide per 100 ml of blood). The total output of blood from the heart in one minute (measured in ml) is given by C, where C is a function of x and y such that $C(x, y) = \frac{100x}{y}$. Find C, when $x = 320$ and $y = 6$?

Solution: The values of $x = 320$ and $y = 6$ are used in the given function to obtain:

$$C(x, y) = (320, 6) = \frac{100(320)}{6} = 5333\,ml \text{ of blood per minute.}$$

The similar procedure can easily be extended to define functions in three, four or more independent variables. Functions of more than one independent variable are called **multivariate functions.**

ii) *Definition of a function of two variables*

Definition 10.1.2:[Limit of a Function]: The notation of a limit of a function of two variables z=f(x,y)

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \qquad (1)$$

means that the functional values f(x, y) can be made arbitrary close to L, by choosing a point (x, y) sufficiently close (but not equal) to the point (x_0, y_0).

Example 10.1.4: Limit of a Function]: Find the limit of a function $f(x, y) = x^2 + y^2 - 3$ at a point $(x_0, y_0) = (2, 4)$.

Solution: By direct substitution, the limit of a function at a point (2, 4) is

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \lim_{(x,y)\to(2,4)} = (x^2 + y^2 - 3) = 4 + 16 - 3 = 17$$

- ***Rules of limits of a function of two variables***

  If the limits of two functions f(x,y) and g(x,y) are

  $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x, y) = M$, L and M are real numbers, then the rules developed are the following:

  Sum rule: $\lim_{(x,y)\to(x_0,y_0)} [f(x, y) + g(x, y)] = L + M$

Difference rule: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)-g(x,y)] = L-M$

Product rule: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)\cdot g(x,y)] = L\cdot M$

Quotient rule: $\lim_{(x,y)\to(x_0,y_0)} \left(\frac{f(x,y)}{g(x,y)}\right) = \frac{L}{M}$

Power rule: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = \left[\lim_{(x,y)\to(x_0,y_0)} f(x,y)\right]^n$

Example 10.1.5: Limit of a Function]: Evaluate the following limits:

a. $\lim_{(x,y)\to(-1,-1)} f(x,y) = \left(x^3+y^2+4x\right)\left(2x^2+3y^2-3xy\right)$

b. $\lim_{(x,y)\to(-1,-1)} f(x,y) = \frac{x^2}{x^2+y^2+2}$

Solution:

a. The limit is related to the product rule:

$$\lim_{(x,y)\to(-1,-1)} f(x,y) = \lim_{(x,y)\to(-1,-1)} \left(x^3+y^2+4x\right) \lim_{(x,y)\to(-1,-1)} \left(2x^2+3y^2-3xy\right)$$
$$=(-1+1-4)(2+3-3)=(-4)(2)=-8$$

b. The limit is related to the quotient rule:

$$\lim_{(x,y)\to(-1,-1)} f(x,y) = \frac{\lim_{(x,y)\to(-1,-1)} x^2}{\lim_{(x,y)\to(-1,-1)} (x^2+y^2+2)} = \frac{1}{1+1+2} = \frac{1}{4}$$

- ***Continuity of a function of two variables***

Definition 10.1.3:[Continuity of a Function]: The function f(x,y) is continuous at a point $p(x_0, y_0)$ if and only if

a. $f(x_0, y_0)$ is defined

b. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

c. The limiting value of a function equals the value of a function at a point $p(x_0, y_0)$:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

Example 10.1.6: Limit of a Function]: Test the continuity of a function

$f(x,y)=\dfrac{x^2+y^2+2xy}{x+y}$ at a point P(2, 2).

Solution: For continuity, we need to test the following three conditions:

1. Condition (1) is satisfied, since the function value at a point p(2,2) is defined:

   $$f(2,2)=\frac{4+4+8}{2+2}=4$$

2. Condition (2) is satisfied, since the limit of a function at a point p(2, 2) exists:

   $$\lim_{(x,y)\to(2,2)} f(x,y)=\lim_{(x,y)\to(2,2)}\frac{x^2+y^2+2xy}{x+y}=4$$

3. Condition (3) is satisfied, since the functional value and the limiting value of a function at a point p(2, 2) are equal:

   The function $f(x,y)=\dfrac{x^2+y^2+2xy}{x+y}$ is therefore declared a continuous function at a point p(2, 2).

(iii) *Definition of partial derivative*

To give clear concept to partial derivative, the problem related to our real-life situations is considered as:

Suppose, a small firm makes only two products, radios and audiocassette recorders. The profit of the firm from these two products is given by:

$$P(x,y)=40x^2-10xy+5y^2-80 \qquad (2)$$

Where x is the number of units of radios sold and y is the number of units of recorders sold. How changes in x will (radios) or y (recorders) affects P (profit)?

Suppose that sales of radios have been steady at 10 units; only the sales of recorders vary. The management would like to find the rate (marginal profit/ derivative of the profit function) at which the y number of recorders sold.

If x is fixed at 10 units, say, then this information reduces the profit two variables function (2) to a new single variable function that can be found from equation (1) by putting x=10:

$$P(10,y)=40(10)^2-10(10)y+5y^2-80=3920-100y+5y^2$$

The function $P(10, y)$ shows the profit from the sales of y recorders, assuming that x is fixed at 10 units. The rate, at which the y number of recorders sold, is the ordinary derivative of P(10,y) with respect to y:

$$\frac{dP(10,y)}{dy} = -100+10y \qquad (3)$$

This represents the per unit profit from y number of audiocassette recorders.

The notation of $dP(10,y)/dy$ is usually stands for **ordinary derivative,** when the function is a single variable function. The profit function (2) is a function of two variables; its rate with respect to y should be a **partial derivative**. For partial derivative with respect to y, the ordinary derivative in equation (3) is replaced by $\partial P(10,y)/\partial y$ to obtain

$$\frac{\partial P}{\partial y} = \frac{\partial P(10,y)}{\partial y} = P_y = -100+10y,$$ prime rotation is not allowed.

Informally,

- the partial derivative $\partial f(x,y)/\partial x$ with respect to x is the derivative of f (x, y) obtained by keeping x as a variable and y as a constant quantity.
- the partial derivative $\partial f(x,y)/\partial y$ with respect to y is the derivative of f(x, y) obtained by keeping y as a variable and x as a constant quantity.

Definition 10.1.4:[Partial Derivatives]: If $z = f(x, y)$ is a function of two variables, then the first partial derivatives of $z = f(x, y)$ with respect to x and y are the functions f_x and f_y respectively, defined by,

$$\frac{\partial f}{\partial x} = f_x(x,y) = f_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \qquad (4)$$

$$\frac{\partial f}{\partial y} = f_y(x,y) = f_y = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \qquad (5)$$

Provided that the limits exist.

iv) *Partial derivatives of a function of two variables*

Example 10.1.7:[Partial Derivatives]: The function is $f(x, y) = x^2y + xy^2$. Find partial derivatives f_x and f_y.

Solution: The partial derivatives of $f(x, y)$ w.r.t x and y are the following :

$$\frac{\partial f}{\partial x} = f_x = \frac{\partial}{\partial x}\left(x^3y + xy^2\right) = 3x^2y + y^2,\ y = constant$$

$$\frac{\partial f}{\partial y} = f_y = \frac{\partial}{\partial y}\left(x^3y + xy^2\right) = x^3 + 2xy,\ x = constant$$

Example 10.1.8:[Partial Derivatives]: The function is $z = x^2 \sin(3y + x^3)$. Find z_x and z_y at a point $(\pi/3, 0)$.

Solution: The partial derivative of z (x, y) w.r.t x is:

$$z_x = \frac{\partial}{\partial x}\left[x^2 \sin(3x + y^3)\right], \quad y = constant$$

$$= 2x\sin(3x + y^3) + x^2\cos(3x + y^3)\frac{\partial}{\partial x}\left(3x + y^3\right)$$

$$= 2x\sin(3x + y^3) + x^2\cos(3x + y^3)(3 + 0)$$

$$= 2x\sin(3x + y^3) + 3x^2\cos(3x + y^3)$$

$$\left(\frac{\partial z}{\partial x}\right)_{(\pi/3,0)} = 2\frac{\pi}{3}\sin\left(\frac{3\pi}{3}\right) + 3\frac{\pi^2}{9}\cos\left(3\frac{\pi}{3}\right)$$

$$= \frac{2\pi}{3}\sin\pi + \frac{3\pi^2}{9}\cos\pi = -\frac{\pi^2}{3}$$

The partial derivative of z(x, y) w.r.t y is:

$$z_y = \frac{\partial}{\partial y}\left[x^2 \sin(3x + y^3)\right], \quad x = constant$$

$$= x^2\frac{\partial}{\partial y}\left[\sin(3x + y^3)\right]$$

$$= x^2\cos(3x + y^3)\frac{\partial}{\partial y}\left(3x + y^3\right)$$

$$= x^2\cos(3x + y^3)(0 + 3y^2) = 3x^2y^2\cos(3x + y^3)$$

$$\left(\frac{\partial z}{\partial y}\right)_{(\pi/3,0)} = 3\left(\frac{\pi^2}{9}\right)(0)^2\cos\left(\frac{3\pi}{3}\right) = 0$$

Example 10.1.9:[Real-Life Problem]: Suppose that the temperature of the water at the point on a river where a nuclear power plant discharges its hot waste water is approximated by

$T(x, y) = 2x + 5y + xy - 40$ (6)

Where x represents the temperature of the river water in degree Celsius before it reaches the power plant and y is the number of megawatts (in hundreds) of electricity being produced by the plant.

a. Find and interpret $T_x(9, 5)$. b. Find and interpret $T_y(9, 5)$.

Solution:

a. The partial derivative of (5) w.r.t x is the rate of change in T with respect to x:

$T_x = 2 + y, \quad y = constant$

This rate with x = 9 and y = 5 is

$[T_x]_{(9,5)} = 2 + y = 2 + 5 = 7$

the approximate change in temperature resulting from a one degree increase in input water, if the input electricity y remains constant at 500 megawatts.

b. The partial derivative of (5) w.r.t y is the rate of change in T with respect to y:

$T_y = 5 + x, \quad x = constant$

This rate with x = 9 and y = 5 is

$[T_y]_{(9,5)} = 5 + x = 5 + 9 = 14$

the approximate change in temperature resulting from a one megawatt increase in production of electricity if the input water temperature x remains constant at $9^0 C$.

Exercise 10.1

1. The function is $f(x, y, z) = x^2 y e^{2x} + (x + y - z)^2$. Find the function value at the following points:

a. $f(0, 0, 0)$ b. $f(1, -1, 1)$ c. $f(-1, 1, -1)$

d. $\frac{\partial}{\partial x} f(x, x, x)$ e. $\frac{\partial}{\partial y} f(1, y, 1)$ f. $\frac{\partial}{\partial z} f(1, 1, z^2)$

2. Find the domain and range of each of the following functions:

a. $f(x, y) = \frac{1}{\sqrt{x - y}}$ b. $f(x, y) = \sqrt{\frac{y}{x}}$

c. $f(x, y) = e^{\frac{x+1}{y-2}}$ d. $f(x, y) = \frac{1}{\sqrt{9 - x^2 - y^2}}$

3. Find the partial derivatives f_x and f_y of each of the following functions:

a. $f(x,y)=\sin(x^2)\cos y$ b. $f(x,y)=\sqrt{3x^2+y^4}$

c. $f(x,y)=xy^3\tan^{-1}y$ d. $f(x,y)=x^3+x^2y+xy^2+y^3$

e. $f(x,y)=\sin^{-1}xy$ f. $f(x,y)=x^2e^{x+y}\cos y$

4. The production function z for the United States was once estimated as:

$z=f(x,y)=x^{0.7}y^{0.3}$

Where x stands for the amount of labor and y stands for the amount of capital. Find the marginal productivity of labor ($\partial z/\partial x$) and of capital ($\partial z/\partial y$).

Hint: Marginal productivity is the rate at which production z changes (increases or decreases) for a unit change in labor x and capital y.

5. A similar production function for Canada is:

$z=f(x,y)=x^{0.4}y^{0.6}$

Where x stands for the amount of labor and y stands for the amount of capital. Find the marginal productivity of labor ($\partial z/\partial x$) and of capital ($\partial z/\partial y$) .

10.2 Euler's Theorem

The specialty of Euler's theorem is to verify the degree of a homogeneous function. The homogeneous function is a function $z=f(x,y)$ not altered if the real numbers x and y of a function $z=f(x,y)$ are stretched or squeezed by any real scalar quantity λ.

i) Definition of homogeneous Function of Degree n

Definition 10.2.1:[Homogeneous Function]: A function $f(x,y)$ is a homogeneous function of degree n in variables x and y, if for all values of the variables and for every positive value of λ, the identity is true:

$$f(\lambda x,\lambda y)=\lambda^n f(x,y) \qquad (7)$$

If we use $\lambda=\frac{1}{x}$, then result (7) becomes:

$$f\left(1,\frac{y}{x}\right)=\left(\frac{1}{x}\right)^n f(x,y)=\frac{1}{x^n}f(x,y) \qquad (8)$$

For example,

$f(x,y)=3x+4y$ is a homogeneous function of degree 1.

$f(x,y)=3x^2+4y^2$ is a homogeneous function of degree 2.

$f(x,y)=3x^2+4xy$ is a homogeneous function of degree 2.

$$f(x, y) = \frac{1}{x+y}\sin\frac{2xy}{x^2+y^2}$$ is a homogeneous function of degree -1

Definition 10.2.2:[Homogeneous Polynomial]: A polynomial function $f(x, y)$ is said to be a homogeneous function of degree n in variables x and y, if the degree of each of its terms in x and y is equal to n, for which the identity is true:

$$\begin{aligned} f(x, y) &= a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_{n-1}xy^{n-1} + a_ny^n \\ &= x^n\left[a_0x^n + a_1(y/x) + a_2(y/x)^2 + \dots + a_{n-1}(y/x)^{n-1} + a_n(y/x)^n\right] \\ &= x^n g\left(\frac{y}{x}\right) \end{aligned} \tag{9}$$

Hence a homogeneous function f(x, y) of degree n is of the form $x^n f(y/x)$, when x > 0.

Example 10.2.1:[Homogeneous Function]: Show that the function

$f(x, y) = 2xy + y^2$ is a homogeneous function of degree 2.

Solution: The function $f(x, y) = 2xy + y^2$ is a homogeneous function of degree 2, if the identity (7) is true:

$$\begin{aligned} f(\lambda x, \lambda y) &= 2(\lambda x)(\lambda y) + (\lambda y)^2 \\ &= 2\lambda^2xy + \lambda^2y^2 \\ &= \lambda^2(xy + y^2) \\ &= \lambda^2 f(x, y),\ n = 2 \end{aligned}$$

Thus, the given function is a homogeneous function of degree 2.

Example 10.2.2:[Homogeneous Polynomial]: Show that a polynomial

$f(x, y) = x^3 - 3x^2y + 6xy^2 + 4y^3$ is a homogeneous polynomial of degree 3.

Solution: The polynomial $f(x, y) = x^3 - 3x^2y + 6xy^2 + 4y^3$ is a homogeneous polynomial of degree 3, if the identity (9) is true:

$$f(x,y)=x^3-3x^2y+6xy^2+4y^3$$
$$=x^3\left(1-3\frac{y}{x}+6\left(\frac{y}{x}\right)^2+4\left(\frac{y}{x}\right)^3\right)$$
$$=x^3g\left(\frac{y}{x}\right),\ n=3$$

Thus, the given polynomial is a homogeneous polynomial of degree 3.

Example 10.2.3:[Homogeneous Function]: Show that the rational function $f(x, y) = \dfrac{\sqrt{y}+\sqrt{x}}{y+x}$ is a homogeneous function.

Solution: The given rational function is a homogeneous function, if the identity (9) is true:

$$f(x,y)=\frac{\sqrt{y}+\sqrt{x}}{y+x}$$
$$=\frac{\sqrt{x}\left(\dfrac{\sqrt{y}}{\sqrt{x}}+1\right)}{x\left(\dfrac{y}{x}+1\right)}$$
$$=\frac{\sqrt{x}}{x}\frac{\left(\left(\dfrac{y}{x}\right)^{1/2}+1\right)}{\left(\left(\dfrac{y}{x}\right)+1\right)}$$
$$=\frac{1}{\sqrt{x}}g\left(\frac{y}{x}\right),\ n=\frac{1}{2}$$

Thus, the given function is a homogeneous function of degree 1/2.

The concept of homogeneity can also be extended to transcendental functions such as in example below.

Example 10.2.4: [Homogeneous Function]: Show that $f(x,y)=\dfrac{1}{x+y}\sin\dfrac{2xy}{x^2+y^2}$ is a homogeneous transcendental function of degree 1.

Solution: The given function is a homogeneous transcendental function of degree 1, if the identity (9) is true:

$$f(x,y)=\frac{1}{x+y}\sin\left(\frac{2xy}{x^2+y^2}\right)$$

$$=\frac{1}{x\left(1+\frac{y}{x}\right)}\sin\frac{x^2\left(2\frac{y}{x}\right)}{x^2\left(1+\left(\frac{y}{x}\right)^2\right)}$$

$$=\frac{1}{x\left(1+\frac{y}{x}\right)}\sin\frac{\left(2\frac{y}{x}\right)}{\left(1+\left(\frac{y}{x}\right)^2\right)}=\frac{1}{x}f\left(\frac{y}{x}\right),\ n=1$$

Thus, the given function is a homogeneous function of degree 1.

ii) Statement of Euler's Theorem based on homogeneous functions

Theorem 10.1: If $z=f(x,y)$ is continuously differentiable and defines a homogeneous function of degree n, then

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz \tag{10}$$

If $z=x^n f(y/x)$, then, its partial derivatives with respect x and y are the following:

$$\frac{\partial z}{\partial x}=nx^{n-1}f\left(\frac{y}{x}\right)+x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$=nx^{n-1}f\left(\frac{y}{x}\right)-yx^{n-2}f'\left(\frac{y}{x}\right) \tag{11}$$

$$\frac{\partial z}{\partial y}=x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)=x^{n-1}f'\left(\frac{y}{x}\right) \tag{12}$$

The addition of the products of (11) by x and (12) by y to obtain the Euler's method of order n:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\left[nx^{n-1}f\left(\frac{y}{x}\right)-yx^{n-2}f'\left(\frac{y}{x}\right)\right]+y\left[x^{n-1}f'\left(\frac{y}{x}\right)\right]$$

$$=nx^{n}f\left(\frac{y}{x}\right)-yx^{n-1}f'\left(\frac{y}{x}\right)+yx^{n-1}f'\left(\frac{y}{x}\right)$$

$$=nx^{n}f\left(\frac{y}{x}\right)$$

$$=nz \qquad (13)$$

The proper proof of the Euler's theorem is beyond of this unit.

iii) *Verification of Euler's Theorem*

Example 10.2.5: [Euler's Theorem]: Use Euler's theorem to verify that the function $z=f(x,y)=ax^2+2bxy+cy^2$ is a homogeneous function of degree 2.

Solution: The homogeneous function and its derivatives

$$z=f(x,y)=ax^2+2bxy+cy^2,\ \frac{\partial z}{\partial x}=2ax+2by,\ \frac{\partial z}{\partial y}=2bx+2cy$$

are used in Euler's result (13) to confirm the degree of homogeneous function:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x(2ax+2by)+y(2bx+2cy)$$

$$=2ax^2+2bxy+2bxy+2cy^2$$

$$=2(ax^2+2bxy+cy^2)$$

$$=2z,\ n=2$$

The Euler's procedure confirmed the second degree of homogeneous function.

Example 10.2.6: [Euler's Theorem]: If $u=\tan^{-1}\dfrac{x^3+y^3}{x-y}$, then, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$$

Solution: The function $u=f(x,y)$ is not a homogeneous function, however, it can be reduced to homogeneous form by introducing a new variable z:

$$z=\tan u$$

$$=\frac{x^3+y^3}{x-y}$$

$$=\frac{x^3\left(1+\left(\frac{y}{x}\right)^3\right)}{x\left(1-\left(\frac{y}{x}\right)\right)}=x^2f\left(\frac{y}{x}\right)$$

The given function is a homogeneous of degree 2. the Euler's theorem in this situation is:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z,\quad n=2 \qquad (14)$$

The partial derivatives of $z=\tan u$ w.r.t x and y

$$\frac{\partial z}{\partial x}=\sec^2 u\frac{\partial u}{\partial x},\quad \frac{\partial z}{\partial y}=\sec^2 u\frac{\partial u}{\partial y}$$

are substituting in (14) to obtain the required result:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=2z,\quad z=\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2z\cos^2 u=2\frac{\sin u}{\cos u}\cos^2 u=2\sin u\cos u=\sin 2u$$

Exercise 10.2

1. Are the following functions homogeneous?

   a. $u=f(x,y)=\sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$? b. $z=f(x,y)=\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$?

   c. $z=f(x,y)=x^3e^{\frac{z}{x}}-3y^2\sqrt{x^2+y^2}$? d. $z=f(x,y)=(x^2+3y^2)^{\frac{1}{3}}$?

2. Verify Euler's theorem for the following homogeneous functions:

   a. $z=f(x,y)=ax^2+2hxy+by^2$ b. $z=f(x,y)=(x^2+xy+y^2)$

   c. $z=f(x,y)=\sin^{-1}\dfrac{x}{y}+\tan^{-1}\dfrac{y}{x}$ d. $z=f(x,y)=\dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$

3. If $u=f(y/x)$, then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0.$

4. If $z=xyf(x/y)$, then show that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=2z.$

5. a. If $u=\tan^{-1}\dfrac{x^2+y^2}{x+y}$, then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\sin u\cos u$

   b. If $u=\sin^{-1}\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0.$

Glossary

- **Function of Two Variables:** A function $z = f(x, y)$ **is a function of two variables x and y, if a unique of z is obtained from each ordered pair of real numbers (x, y).** The **real numbers** x and y are **independent variables**; z is the **dependent variable.** The set of all ordered pairs of real numbers (x, y) such that $f(x, y)$ is a real number, is **the domain** of f and the set of all values of $f(x, y)$ is the **range.**

- **Polynomial Function:** A polynomial function in x and y is the sum of functions of the form $f(x, y) = Cx^m y^n$

- **Limit of a Function of two variables :** The notation of a limit of a function of two variables $z = f(x, y)$

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

- Continuity: **The function $f(x, y)$ is continuous at the point (x_o, y_o) if and only if**
  - $f(x_0, y_0)$ is defined
  - $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists
  - $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$

- **Partial Derivatives:** If $z = f(x, y)$ is a function of two variables, then the first partial derivatives of $z = f(x, y)$ with respect to x and y are the functions f_x and f_y respectively, defined by,

$$f_x(x, y) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}, \quad f_y(x, y) = \lim_{\Delta x\to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

- **Homogeneous Function:** A function $f(x, y)$ is a homogeneous function of degree n in variables x and y, if for all values of the variables and for every positive value of λ, for which the identity is true:

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

- **Euler's Theorem:** The specialty of Euler's theorem is to verify the degree of a homogeneous function. The homogeneous function is a function $z = f(x, y)$ not altered if the real numbers x and y of a function $z = f(x, y)$ are stretched or squeezéd by any real scalar quantity λ.

UNIT 11

INTRODUCTION TO NUMERICAL METHODS

This unit tells us, how to:

- describe the importance of numerical methods.
- explain the basic principles of solving a single variable non-linear equations.
- calculate the approximate roots of a single variable non-linear equations by bisection method, regula-falsi method and Newton-Raphson method.
- define the numerical quadrature.
- use the numerical methods such as Trapezoidal rule and Simpson's rule to compute the approximate value of a definte integral without error terms.

11.1 Numerical Solution of Non-linear Equations

Scientists, economists, engineers, an other researchers study relationships between quantities. For example, an engineer may need to know how the illumination from a light source on an object is related to the distance between the object and the source; a biologist may wish to investigate how the population of a bacterial colony varies with time in the presence of a toxin; an economist may wish to determine the relationship between demand for a certain commodity and its market price. The mathematical study of such relationships involves the concept of non-linear equations. For example, the value of the assets of a certain company at time t years is modeled by a non-linear equation $f(t)=100,000-75,000e^{-.2t}$ where t is measured in years. The standing rule for solving non-linear equations algebraic is quadratic formula. In this case, it is not valid to obtain the actual number of t (years) at which the asset function f(t) is going to be zero. Now we are in position to obtain the approximate number of year's t that can be found by using some numerical procedures. For this unit, the numerical procedures recommended are the bracketing methods and iterative methods.

i) Importance of numerical methods

Numerical analysis is the theory of constructive methods in mathematical analysis. Constructive methods in their turn mean a procedure that permits us to obtain the solution of a mathematical problem with an arbitrary precision in a finite number of steps that can be prepared rationally (the number of steps depending on the desired accuracy).

Numerical analysis is both a Science and an Art. As a Science, it is concerned

with the process by which a mathematical problem can be solved arithmetically. As an Art, numerical analysis is concerned with choosing that procedure which is best suited to the solution of a particular problem.

Students of numerical analysis should have the following objectives in view. First, he should obtain an intuitive and working understanding of some numerical methods for the basic problems of numerical analysis. Second, he should gain some appreciation of the concept of error and of the need to analyze and predict it. Third, he should develop some experience in the implementation of numerical method by using **computer software**. In this unit, emphasis is placed on understanding why numerical methods work and their limitations.

ii) Basic principles for solving non-linear equations

Definition 11.1.1: [Root of an Equation, Zero of a Function]: If f(x) is any **continuous function** of a single variable x, then any number r for which f(r) = 0 is called a **root** of f(x) = 0. Also we say that r is a zero of the function f(x).

If f(x) is any algebraic function (non-linear equation), then the actual roots of f(x) can be found by direct rules, such as quadratic formula, common factors procedure and synthetic division.

For example, the quadratic equation $x^2+5x+6=0$ (or second degree non-linear equation or algebraic equation) has two **actual** (or exact) roots $r_1=-2$ and $r_2=-3$ obtained by common factors procedure/quadratic formula:

$$f(x)=x^2+5x+6=0$$
$$=x^2+2x+3x+6=0$$
$$=(x+2)(x+3)=0 \Rightarrow x+2=0,\ x+3=0 \Rightarrow x=-2=r_1,\ x=-3=r_2$$

On the other hand, the actual roots of nonalgebraic equation (or transcendental equation) $x^{-2}+5x+6=0$ are not possible by applying quadratic formula. The only way is to find out the **approximate roots** that can be found by using some **numerical procedures**. The numerical procedures are the bracketing methods and iterative methods.

iii) Calculation of approximate roots of non-linear equations

For approximate roots of a non-linear equations, the numerical procedures **Bisection method** and **regula-falsi method** are the bracketing methods that depend on two initial approximations that must be in the shape of a closed interval [a, b].

The non-linear curve f(x) has the function values f(a) and f(b) in the interval [a, b] that must be opposite in signs for showing its continuity. Once the interval has been found, no matter how large, the **iteration** will be preceded until an approximate root is obtained.

The fundamental principle in computer science is the **iteration.** As the name suggests, it means that a process of bisection or regula-falsi method is repeated unit an answer is achieved.

- ***Bisection Method***

If y= $f(x)$ is continuous function in the interval [a,b], then, it will cross the x-axis at a point (r,0) whose x-coordinate x = r will keep as actual root that lies somewhere in the interval [a, b]. This is shown in the fig.11.1 :

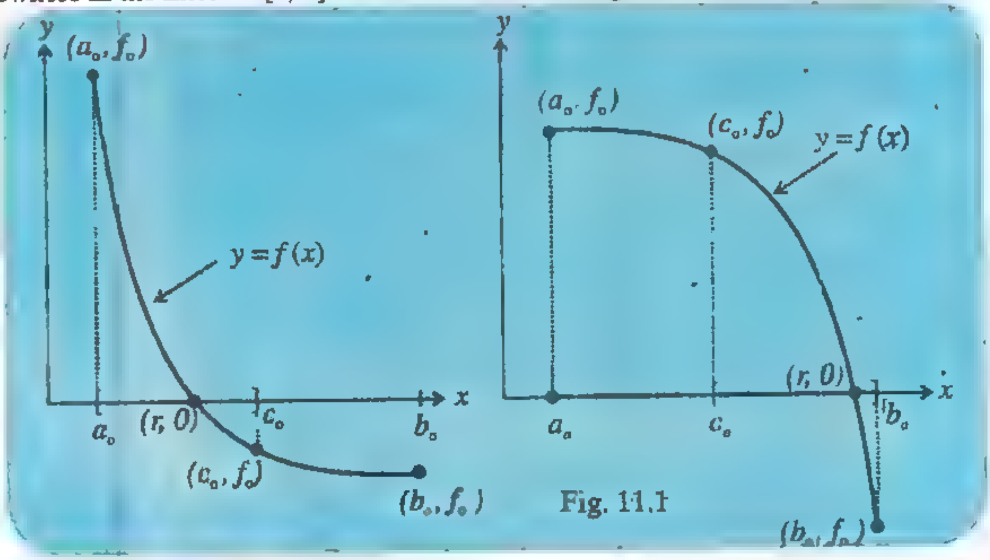


Fig. 11.1

The bisection method systematically moves the endpoints of the interval [a,b] closer and closer together till it reaches an interval of small width that brackets the root r. The decision step for this process of interval halving is to choose the midpoint $c = (a+b)/2$ and then analyze the three possibilities that might arise:

1. If the function values $f(a)$ and $f(c)$ at $x = a$ and $x = c$ have opposite signs, then the approximate root lies in interval [a, c] and discard b.
2. If the function values $f(c)$ and f(b) at $x = c$ and $x = b$ have opposite signs, then the approximate root lies in interval [c, b] and discard a.

3. If the function value at $x = c$ is $f(c) = 0$, then c is our approximate root to actual root r.

If either of cases 1 or 2 occurs, we have an interval half as wide as the original interval that contains the root, and we are "squeezing down on it" (see Fig. 11.1). To continue the process, relabel the new smaller interval and repeat the sequence of nested intervals and their midpoints.

The procedure in detail is as under:

The given interval $[a_0, b_0]$ is the initial interval at which the function f(x) must be opposite in signs. At this stage, the initial interval brackets the actual root r whose midpoint is $c_0 = (a_0 + b_0)/2$. It develops the first iterate c_o (through properties 1 or 2) to actual root r in the interval $[a_0, b_0]$.

The next interval $[a_1, b_1]$ is the first interval which brackets the actual root r and c_1 is its midpoint $c_1 = (a_1 + b_1)/2$. It develops the second iterate c_1 (through properties 1 or 2) to actual root r in the interval $[a_1, b_1]$.

Similarly, the interval $[a_n, b_n]$ is the nth interval which brackets the actual root r and c_n is its midpoint $c_n = (a_n + b_n)/2$. It develops the (n-1)-th iterate c_{n-1} to actual root r in the n-th interval $[a_n, b_n]$.

This completes the n times iteration of the bisection method and the midpoint c_{n-1} is taken as the desired approximation to the actual root $r \approx c_{n-1}$ of $y = f(x)$.

Bisection Method: To find approximate root of an equation $f(x) = 0$ in the interval [a, b]. Proceed with the method only if f (x) is continuous and $f(a)$ and f (b) have opposite in signs.

Example 11.1.1: [Bisection method]: Perform two iterations of the bisection method to approximate the actual root r of the non-linear equation $f(x) = \sin x - e^{-x}$ (x is in radians) in the interval [0.5, 0.7].

Solution: Reset the given interval to obtain the initial interval $[a_0, b_0] = [0.5, 0.7]$ and compute the function $f(x) = \sin x - e^{-x}$ values at $x = a_0 = 0.5$ and $x = b_o = 0.7$ to obtain:

$$\left.\begin{aligned} f(a_o) = f(0.5) = \sin 0.5 - e^{-0.5} &= -0.127 < 0 \\ f(b_o) = f(0.7) = \sin 0.7 - e^{-0.7} &= +0.148 > 0 \end{aligned}\right\} \textit{opposite in signs}$$

The function values $f(a_o)$ and $f(b_o)$ are opposite in signs, so the actual root r of f (x) lies in the interval [0.5,0.7].

1. The midpoint of the initial interval $[a_0, b_0]$ is $c_o = \frac{a_o + b_o}{2} = \frac{0.5+0.7}{2} = 0.6$, and the function values at $x = a_0 = 0.5$, $x = b_0 = 0.7$, $c_0 = 0.6$ are the following:

$$\left.\begin{aligned} f(a_o) &= f(0.5) = \sin 0.5 - e^{-0.5} = -0.127 < 0 \\ f(b_o) &= f(0.7) = \sin 0.7 - e^{-0.7} = +0.148 > 0 \\ f(c_o) &= f(0.6) = \sin 0.6 - e^{-0.6} = +0.016 > 0 \end{aligned}\right\}$$

The function values $f(a_o)$ and $f(c_o)$ are opposite in signs, so the approximation to the actual root r of f (x) lies in the interval [0.5, 0.6] and discard $b_0 = 0.7$. The initial interval is reset to obtain the first interval $[a_1, b_1] = [0.5, 0.6]$.

2. The midpoint of the first interval is $c_1 = \frac{a_1 + b_1}{2} = \frac{0.5+0.6}{2} = 0.55$, and the function values at $x = a_1 = 0.5$, $x = b_1 = 0.6$, and $x = c_1 = 0.55$ are the following:

$$\left.\begin{aligned} f(a_1) &= f(0.5) = \sin 0.5 - e^{-0.5} = -0.127 < 0 \\ f(c_1) &= f(0.55) = \sin 0.55 - e^{-0.55} = -0.054 < 0 \\ f(b_1) &= f(0.6) = \sin 0.6 - e^{-0.6} = +0.016 > 0 \end{aligned}\right\}$$

The function values $f(b_1)$ and $f(c_1)$ are opposite in signs, so the approximation to actual root r of f (x) lies in the interval [0.55, 0.6] and discard $a_1 = 0.5$. The first interval is reset to obtain the second interval $[a_2, b_2] = [0.55, 0.6]$.

After second iteration of the bisection method, the midpoint $c_1 = 0.55$ is declared approximate root to actual root $r \approx c_1$. The approximation value of a function $f(x) = \sin x - e^{-x}$ at approximate root $r \approx 0.55$ is $f(0.055) = -0.054$.

- ***Regula-Falsi method***

The next bracketing method is the method of regula-falsi method. It was developed because the bisection method converges at a fairly slow speed. As before, we assume that $f(a)$ and $f(b)$ have opposite signs. The bisection method always used

we assume that $f(a)$ and $f(b)$ have opposite signs. The bisection method always used the midpoint of the interval as the next iterate, but in regula-falsi method, the next iterate is anywhere in the interval [a, b] represented by the point of intersection (c, 0) of the straight line formed by the points $(a, f(a))$, and $(b, f(b))$ and the x-axis

$$\left.\begin{aligned} \frac{y-f(a)}{x-a} &= \frac{f(b)-f(a)}{b-a} \\ \frac{0-f(a)}{c-a} &= \frac{f(b)-f(a)}{b-a} \\ c &= a-\frac{(a-b)f(a)}{f(a)-f(b)} \end{aligned}\right\}, \ at\ (x, y)=(c, 0) \qquad (1)$$

and then analyze the three possibilities that might arise :

1. If the function values $f(a)$ and $f(c)$ at x=a and x=c have opposite signs, then the root lies in the interval [a, c] and discard b.
2. If the function values $f(c)$ and f(b) at x=c and x=b have opposite signs, then the root lies in the interval [c, b] and discard a.
3. If the function value at x=c is $f(c) = 0$, then c is our approximate root.

This is shown in the Fig. 11.2.

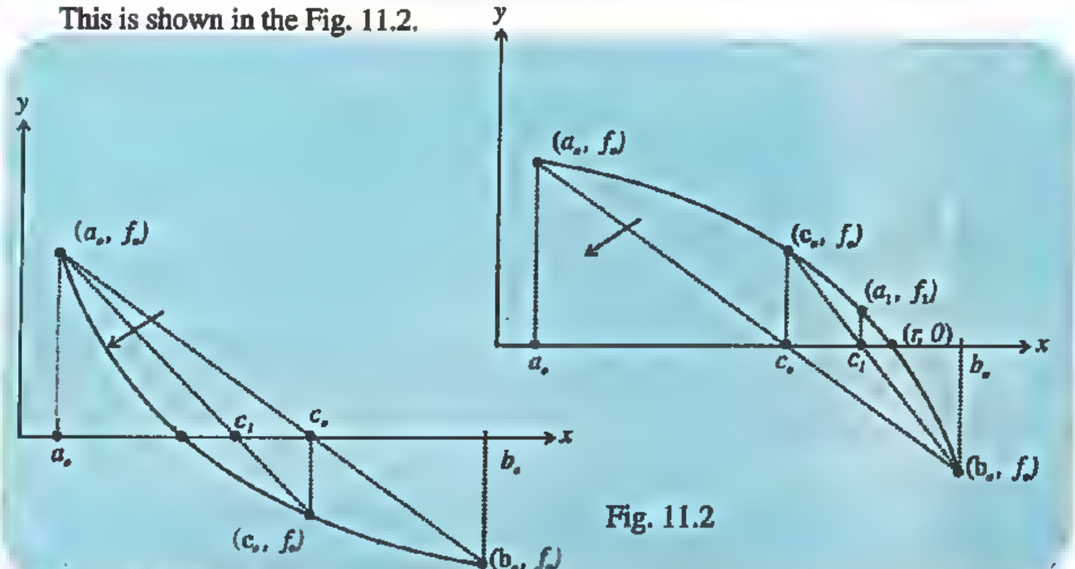


Fig. 11.2

Regula-Falsi Method: To find approximate root of an equation f (x)=0 in the interval [a, b]. Proceed with the method only if f(x) is continuous and f (a) and f (b) have opposite in signs.

Example 11.1.2:[Regula-Falsi method]: Perform two iterations of the regula-falsi method to approximate the actual root r of the non-linear equation $f(x)=\sin x-e^{-x}$ (x is in radians) in the interval [0.5, 0.7].

Solution: Reset the given interval to obtain the initial interval $[a_0,b_0]=[0.5,0.7]$ and compute the function $f(x)=\sin x-e^{-x}$ values at $x=a_0=0.5$ and $x=b_o=0.7$ to obtain:

$$\left.\begin{aligned} f(a_o)&=f(0.5)=\sin 0.5-e^{-0.5}=-0.127<0\\ f(b_o)&=f(0.7)=\sin 0.7-e^{-0.7}=+0.148>0\end{aligned}\right\} \textit{opposite in signs}$$

The function values $f(a_o)$ and $f(b_o)$ are opposite in signs, so the actual root r of f (x) lies in the interval [0.5,0.7].

Equation (1) is used to obtain

$$c_o=a_o-\frac{(a_o-b_o)f(a_o)}{f(a_o)-f(b_o)}=0.5-\frac{(0.5-0.7)(-0.127)}{-0.127-0.148}=0.592364$$

That provides the function value at c_0=0.592364:

$$f(c_o)=\sin(0.592364)-e^{-0.592364}=+0.081$$

1. The function values at $x=a_0=0.5$, $x=b_0=0.7$, $c_0=0.592364$ are the following:

$$\left.\begin{aligned} f(a_o)&=\sin 0.5-e^{-0.5}=-0.127<0\\ f(b_o)&=\sin 0.7-e^{-0.7}=+0.148>0\\ f(c_o)&=\sin 0.592364-e^{-0.592364}=+0.0053>0\end{aligned}\right\}$$

The function values $f(a_o)$ and $f(c_o)$ are opposite in signs, so the approximation to actual root r of f (x) lies in the interval [0.5, 0.592364] and discard $b_0=0.7$. The initial interval is reset to obtain the first interval$[a_1,b_1]$=[0.5,0.592364].

Equation (1) is used to obtain:

$$c_1=a_1-\frac{(a_1-b_1)f(a_1)}{f(a_1)-f(b_1)}=0.5-\frac{(0.5-0.592364)(-0.127)}{-0.127-0.0053}=0.588662$$

The function value at $c_1=0.588662$ is:

$$f(0.588662)=\sin(0.588662)-e^{-0.588662}=0.00018$$

2. The function values at $x=a_1=0.5$, $x=b_1=0.592364$, $c_1=0.588662$ are the following:

$$\left.\begin{aligned} f(a_1)&=\sin 0.5-e^{-0.5}=-0.127<0\\ f(b_1)&=\sin 0.592364-e^{-0.592364}=+0.081>0\\ f(c_1)&=\sin 0.588662-e^{-0.588662}=+0.00018>0\end{aligned}\right\}$$

The function values $f(a_1)$ and $f(c_1)$ are opposite in signs, so the approximation to the actual root r of f (x) lies in the interval [0.5, 0.588662] and discard $b_1=0.592364$. The first interval is reset to obtain the second interval $[a_2, b_2]=[0.5, 0.588662]$.

After second iteration of the regula-falsi, the point $c_1=0.588662$ is declared as approximation to the actual root $r\approx c_1$. The approximate value of a function $f(x)=\sin x-e^{-x}$ at approximate root $c_1=0.588662$ is $f(0.588662)=0.00018$.

• *Newton-Raphson-method*

Another numerical procedure is the Newton-Raphson method under the umbrella of **iterative methods.** It was first used by Newton and then was refined and improved by joseph Raphson (1648-1715). The Newton-Raphson method is one-point **iterative method** (not bracketing method) which requires **one** previous approximation in contrast of bracketing methods which require **two** in computing the successive approximation.

The Newton-Raphson method uses the **slopes of the tangent lines to the graph of a function f(x)** to approximate roots of the equation $f(x)=0$.

If $f(x)$ and $f'(x)$ are continuous near actual roots, then this extra information regarding the nature of $f(x)$ can be used to develop a sequence of iterates $\{x_n\}$ that will converge faster to actual root than either the bisection and regula-falsi methods.

If r is any actual root of an equation of the form $f(x)=0$, and x_0 is an initial approximation to the actual r, then the tangent line on a curve f(x) at a point (x_0, f_0) crosses the x-axis at a point $(x_1, 0)$. The point $(x_1, 0)$ is the point of intersection of the tangent line and the x-axis, and the x-coordinate x_1 of the point of intersection $(x_1, 0)$ is our first approximation to the actual root r of an equation $f(x)=0$. This is shown in the Fig. 11.3.

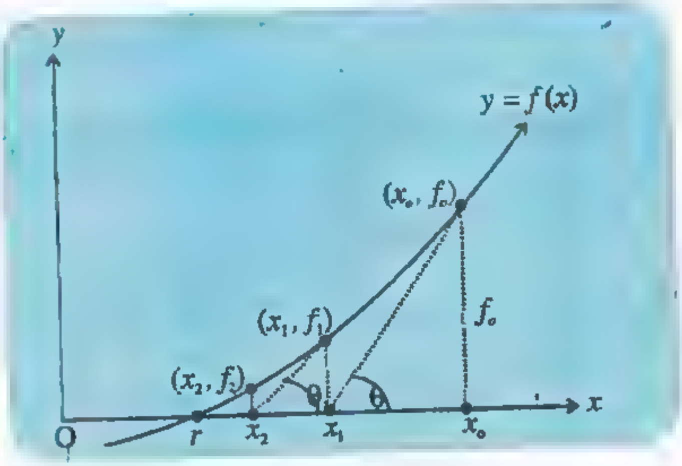


Fig. 11.3

The slope of the tangent line on a curve $y=f(x)$ at a point (x_0, f_0) is used to obtain the first approximation (iterate) x_1 :

$$\left.\begin{aligned} \tan\theta &= \frac{f(x_o)}{x_o - x_1} \\ f'(x_o) &= \frac{f(x_o)}{x_o - x_1} \end{aligned}\right\} \Rightarrow x_o - x_1 = \frac{f(x_o)}{f'(x_o)} \Rightarrow x_1 = x_o - \frac{f(x_o)}{f'(x_o)} \qquad (2)$$

Similarly, the slope of the tangent line on a curve $y=f(x)$ at a point (x_1, f_1) is used to obtain the second iterate x_2 :

$$\left.\begin{aligned} \tan\theta_1 &= \frac{f(x_1)}{x_1 - x_2} \\ f'(x_1) &= \frac{f(x_1)}{x_1 - x_2} \end{aligned}\right\} \Rightarrow x_1 - x_2 = \frac{f(x_1)}{f'(x_1)} \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

This procedure of slope finding method is continued till it reaches the (i+1)-th iterate:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)},\ i = 0, 1, 2, \ldots \qquad (3)$$

which is called the **Newton-Raphson method.**

Newton-Raphson Method: To find approximate root of an equation of the form f(x)=0 with initial iterate x_0, the iteration method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad i=0,1,2,3,\ldots, \quad i=1,2,3,\ldots$$

develops a sequence of successive iterates $\{x_n\}$ that will converge faster to actual root r than either the bisection and regula-falsi methods.

Example:11.1.3:[Newton-Raphson method]: Use Newton-Raphson iterative method to approximate the actual root $r = 0.438447$ of the non-linear equation $f(x) = x^2 - 5x + 2$ with initial start $x_0 = 0.4$ that must be accurate to six decimal places.

Solution: The given non-linear equation $f(x) = x^2 - 5x + 2$ and its derivative $f'(x) = 2x - 5$ are used in the Newton-Raphson iterative method (3) to obtain the successive iterates

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad i = 0,1,2$$

$$x_1 = x_o - \frac{f(x_o)}{f'(x_o)}$$

$$= \frac{x_o^2 - 2}{2x_o - 5} = \frac{(0.4)^2 - 2}{2(0.4) - 5} = 0.438095, \quad i = 0$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= \frac{x_1^2 - 2}{2x_1 - 5} = \frac{(0.438095)^2 - 2}{2(0.438095) - 5} = 0.438447, \quad i = 1$$

which converges to the actual root $r = 0.438447$.

The second iterate $x_2 = 0.438447$ agrees to six decimal accuracy of actual root $r = 0.438447$. We achieved the six decimal accuracy in just two iterates of the Newton-Raphson method.

Example:11.1.4:[Newton-Raphson method]: Find $\sqrt{5}$, (the square roots of any positive quantity, say 5) by Newton-Raphson iterative method with initial start $x_0 = 2$.

Solution: For square roots of 5, the function must be a quadratic function $f(x_i) = x_i^2 - 5$ and its derivative w.r.t x is $f'(x_i) = 2x_i$

The given function and its derivative are used in Newton-Raphson iterative method (3) to obtain the successive iterates

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)},\ i = 0,1,2$$

$$x_1 = x_o - \frac{x_o^2 - 5}{2x_o} = \frac{x_o^2 + 5}{2x_o} = \frac{(2)^2 + 5}{2(2)} = 2.25,\ \ i = 0$$

$$x_2 = x_1 - \frac{x_1^2 - 5}{2x_1} = \frac{x_1^2 + 5}{2x_1} = \frac{(2.25)^2 + 5}{2(2.25)} = 2.236111111,\ \ i = 1$$

$$x_3 = x_2 - \frac{x_2^2 - 5}{2x_2} = \frac{x_2^2 + 5}{2x_2} = \frac{(2.36111111)^2 + 5}{2(2.36111111)} = 2.236067978,\ \ i = 2$$

which converges to the actual value of $\sqrt{5} = 2.236067978$. (obtained by calculator).

The third iterate $x_3 = 2.236067978$ agrees to nine decimal accuracy of actual root $\sqrt{5} = 2.236067978$. We achieved the nine decimal accuracy in just three iterates of the Newton-Raphson method.

Example: 11.1.5:[Newton-Raphson method]: Use Newton-Raphson method to approximate actual root r of an equation of the form $x^3 + x + 1 = 0$ in the interval [-2, 2].

Solution: The given non-linear equation $f(x) = x^3 + x + 1$ and its derivative $f'(x) = 3x^2 + 1$ are used in Newton-Raphson iterative method (3) to obtain:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^3 + x_i + 1}{3x_i^2 + 1} = \frac{2x_i^3 - 1}{3x_i^2 + 1},\ \ i = 0,1,2,\ldots$$

A convenient choice for the initial iterates is $x_0 = -1$. Use this initial iterate $x_0 = -1$ in the above equation to obtain the sequence of successive iterates

$$x_{i+1} = \frac{2x_i^3 - 1}{3x_i^2 + 1},\ \ i = 0,1,2,\ldots$$

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 + 1} = \frac{2(-1)^3 - 1}{3(-1)^2 + 1} = -0.75\ \ i = 0$$

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 + 1} = \frac{2(-0.75)^3 - 1}{3(-0.75)^2 + 1} = -0.6860465,\ \ i = 1$$

$$x_3=\frac{2x_2^3-1}{3x_2^2+1}=\frac{2(-0.6860465)^3-1}{3(-0.6860465)^2+1}=-0.6823396,\ i=2$$

that converges to the actual root r of the given equation.

Exercise 11.1

1. Find an interval $a \le x \le b$ at which $f(a)$ and $f(b)$ have opposite signs for the following functions:
   a. $f(x)=e^x-2-x$
   b. $f(x)=\cos x+1-x$ x is in radians
   c. $f(x)=x^2-10x+23$
2. Compute four iterates of the bisection method for the following functions with indicated interval $[a_0,b_0]$:
   a. $f(x)=e^x-2-x$, $[1, 1.8]$
   b. $f(x)=\cos x+1-x$, $[0.8,\ 1.6]$ x is in radians
   c. $f(x)=x^2-10x+23$ $[3.2,\ 4]$
3. Compute four iterates of the regula-falsi method for the following functions with indicated interval $[a_0,b_0]$:
   a. $f(x)=e^x-x$, $[-2.4,\ -1.6]$
   b. $f(x)=\cos x+1-x$, $[0.8,\ 1.6]$ x is in radians
   c. $f(x)=x^2-10x+23$, $[-2.4,\ -1.6]$
4. Let $f(x)=x^2-2x-1$. The actual roots of f(x) are r_1=2.414214 and $r_2=1.414214$. Use Newton-Raphson method
   a. with the initial start $x_o=2.5$ to compute the iterates x_1, x_2 and x_3 that will converge the actual root r_1 of f(x).
   b. with the initial start $x_o=-0.5$ to compute the iterates x_1, x_2 and x_3 that will converge the actual root r_2 of f(x).
5. Find iterate x_3 of Newton-Raphson iterative method for the following functions with initial start x_o:
   a. $f(x)=x^3-3$, $x_o=1$
   b. $f(x)=\sin(x), x_o=1$

c. $f(x)=x^3+2x-1,\ x_o=0$

6. Use Newton-Raphson iterative method to approximate the actual r of the following non-linear equations with indicated interval:

a. $f(x)=x^3+3x-1=0$ on (0, 1)

b. $f(x)=x^3+2x^2-x+1=0$ on (-3, -2)

c. $\sqrt[3]{x-3}=x+1$ on [-3, -2]

Continue the process until two consecutive iterates will agree to three decimal places.

7. Can Newton-Raphson iterative method be used for the following functions?

a. $f(x)=9x^4-16x^3-36x^2+96x-60=0,\ x_o=\frac{4}{3}$

b. $f(x)=1-\frac{1}{x},\ x_o=2$

11.2 Numerical Quadrature

Numerical integration is a primary tool used by engineers and scientists to obtain approximate solutions for definite integrals that cannot be solved analytically. For example, the integral

$$I=\int_0^1 e^{x^2}dx \tag{4}$$

has no actual solution. By any integral formula we are not in position to evaluate the given integral (4) for obtaining the actual solution directly. The only way is to find out the approximate solution that can be found by using some numerical procedures, such, as numerical integration.

We now approach the subject of numerical integration. The goal is to approximate the definite integral of $f(x)$

$$I=\int_a^b f(x)dx \tag{5}$$

over the interval [a, b] by evaluating $f(x)$ at a finite number of equally spaced grid points:

$$\left.\begin{array}{llllll} x: & a=x_o & x_1 & \dots & x_{n-1} & x_n=b \\ f(x): & f_o & f_1 & \dots & f_{n-1} & f_n \end{array}\right\}$$

i) *Definition of numerical quadrature*

Definition 11.2.1: [Numerical Quadrature]: If a set of points in the interval $[a, b]$ is $a=x_o<x_1<x_2<\dots<x_{n-1}<x_n=b$, then an expression of the form

$$Q[f(x)]=\sum_{j=0}^{n} w_j f(x_j)=w_o f(x_0)+w_1 f(x_1)+\dots+w_n f(x_n)$$

with the property

$$\int_a^b f(x)dx=Q[f(x)]+E[f(x)] \qquad (6)$$

is called a numerical integration or **quadrature formula**. The term $E[f(x)]$ is called the truncation error for integration. The values $\{x_j\}_{j=0}^{n}$ are called the **quadrature nodes** and $\{w_j\}_{j=0}^{n}$ are called the **weights.**

Depend on the given numerical procedure, the grid points $\{x_j\}$ are chosen in various ways. For trapezoidal rule and Simpson's rule, the grid points are chosen to be equally spaced. In this unit, the numerical procedures recommended are the trapezoidal rule and Simpson's rule.

- ***Approximation by rectangles***

If $f(x)=0$ is a function over the interval $[a, b]$, then the definite integral (5) represents the actual area under the graph of $f(x)$ on the interval $[a, b]$. This is shown in the Fig. 11.4

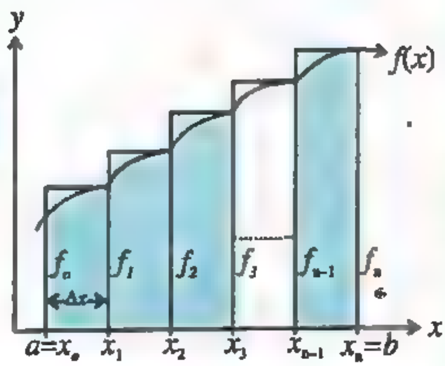


Fig. 11.4

For approximate area, the function $f(x)$ must be known at equally spaced grid points in the interval$[a, b]$, each of width $\Delta x=(b-a)/n$:

$$\left.\begin{array}{llllll} x: & a=x_o & x_1 & x_2\dots\dots x_{n-1} & x_n=b \\ f(x): & f_o & f_1 & f_2\dots\dots f_{n-1} & f_n \end{array}\right\}, \; x_k=x_o+k\Delta x, \; k=0, 1, 2, \dots, n$$

In light of above equally spaced grid points, the actual area (5) under a curve f(x) over the interval [a, b] is rearranged as under:

$$I=\int_a^b f(x)dx=\int_{x_0}^{x_1} f(x)dx+\int_{x_1}^{x_2} f(x)dx+\ldots+\int_{x_{n-1}}^{x_n} f(x)dx \qquad (7)$$

Approximate Area by Rectangles: Consider the initial subinterval $[x_0, x_1]$. Let x_1 denote the right endpoint of the initial subinterval, and the base of the rectangle is of course the initial subinterval and its height is $f(x_0)$. The area of the rectangle in the initial subinterval is therefore $f(x_0)\Delta x$.

If $\int_{x_0}^{x_1} f(x)dx$ is the actual area in the initial subinterval, then the approximate area $f(x_0)\Delta x$ is of course the area of the rectangle lies in the initial subinterval $[x_0, x_1]$. Thus, the sum of the areas of all n rectangles is giving approximate area under the curve f(x) to actual area represented by definite Integral (7):

$$I=\int_a^b f(x)dx\approx R_n$$
$$=f(x_0)\Delta x+f(x_1)\Delta x+f(x_2)\Delta x+\ldots+f(x_n)\Delta x \qquad (8)$$

This approximation improves as the number of rectangles increases, and we can estimate the integral to any desired degree of accuracy by taking n large enough. However, because fairly large values of n are usually required to achieve reasonable accuracy, approximation by rectangles is rarely used in practice.

- ***The Trapezoidal rule***

The accuracy of the approximation can be improved if trapezoids are used instead of rectangles. Figure (11.5) shows the area approximated by n trapezoids instead of n rectangles.

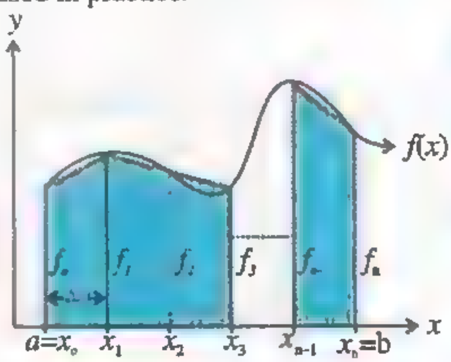


Fig. 11.5

If $\int_{x_0}^{x_1} f(x)dx$ is the actual area in the initial subinterval, then the approximate area $\left[\dfrac{f(x_0)+f(x_1)}{2}\right]\Delta x$ is of course the area of the trapezoid lies in the initial subinterval $[x_0, x_1]$. Thus, the sum of the areas of all n

trapezoids is giving approximate area under the curve f(x) to actual area represented by definite integral (7):

$$\int_a^b f(x)dx \approx T_n = \frac{1}{2}\left[f(x_o)+f(x_1)\right]\Delta x+\frac{1}{2}\left[f(x_1)+f(x_2)\right]\Delta x+........$$

$$+\frac{1}{2}\left[f(x_{n-1})+f(x_n)\right]\Delta x$$

$$=\frac{1}{2}\left[f_0+2f_1+2f_2+.......+2f_{n-1}+f_n\right]\Delta x \qquad (9)$$

Trapezoidal Rule: If $f(x)$ is continuous on $[a, b]$, then the trapezoidal rule is

$$I=\int_a^b f(x) \approx T_n = \frac{\Delta x}{2}\left[f_0+2f_1+2f_2+......+2f_{n-1}+f_n\right],$$

The n^{th} subinterval is $x_n = x_o + n\Delta x \Rightarrow b = a + n\Delta x$ that gives $\Delta x = (b-a)/n$. Moreover, the larger the value of n, the better the approximation.

Example 11.2.1: [Trapezoidal Rule]: Approximate the definite integral $I=\int_{-1}^{2} x^2 dx$ in n=4 subintervals and then compare your approximate answer with the actual value of the definite integral that must be accurate to 5 decimal places.

Solution: The given interval is $[a, b] = [-1, 2]$ and the width for n = 4 subintervals is

$$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{4} = \frac{3}{4} = 0.75$$

The trapezoidal rule (9) is used for $\Delta x = 0.75$ and n=4 to obtain:

$$\int_{-1}^{2} x^2 dx \approx T_4 = \frac{1}{2}\left[f_0+2f_1+2f_2+2f_3+f_4\right](0.75) \qquad (10)$$

The function values f_0, f_1, f_2, f_3, f_4 at grid points x_0, x_1, x_2, x_3, x_4 are

$x_o = a = -1,$ $\qquad f_0 = f(-1) = (-1)^2 = 1$

$x_1 = a + 1\Delta x = -1 + \frac{3}{4} = -\frac{1}{4},$ $\qquad f_1 = \left(-\frac{1}{4}\right)^2 = \frac{1}{16} 0.0625$

$x_2 = a + 2\Delta x = -1 + 2\left(\frac{3}{4}\right) = \frac{1}{2},$ $\qquad f_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25$

$$x_3 = a+3\Delta x = -1+3\left(\frac{3}{4}\right)=\frac{5}{4}, \qquad f_3=\left(\frac{5}{4}\right)^2=\frac{25}{16}=0=1.5625$$

$$x_4 = a+4\Delta x = -1+4\left(\frac{3}{4}\right)=2, \qquad f_4=(2)^2=4$$

used in (10) to obtain:

$$\int_{-1}^{2} x^2 dx \approx T_4 = \frac{1}{2}[f_0+2f_1+2f_2+2f_3+f_4]\Delta x$$
$$=\frac{1}{2}\left[1+2(0.0625)+2(0.25)+2(1.5625)+4\right](0.75)$$
$$=3.28125$$

The exact value of the definite integral is:

$$I=\int_{-1}^{2} x^2 dx = \left|\frac{x^3}{3}\right|_{-1}^{2} = \frac{8}{3}+\frac{1}{3}=\frac{9}{3}=3$$

The trapezoidal approximation T_4 developed an error, which we denote by E_4:

$$E_4 = Actual - Approximation = I - T_4 = 3-3.28125=-0.28125$$

The negative sign indicates that the trapezoidal formula overestimated the true value of the definite integral.

In terms of numerical quadrature notation, $Q\left[f(x)\right]=T_4=3.28125$ and $E_4\left[(x)\right]=-0.28125$ with nodes $x_o=-1,\ x_1=-1/4,\ x_2=1/2,\ x_3=5/4,\ x_4=2$.

- ***The Simpson's rule***

The accuracy of the approximation can be improved if instead of trapezoidal strips, the parabolic strips are used in three consecutive equally spaced grid points. The name given to the procedure in which the approximating strip has a parabolic arc is the **Simpson's rule.**

Mathematically, the parabolic arc is represented by a second degree polynomial $p(x)=Ax^2+Bx+C$.

Simpson's rule approximates the actual area in an interval [a, b] by parabolic arc if f(x) is replaced by second degree polynomial p(x)

$$\int_a^b f(x)dx \approx \int_a^b p(x)dx$$
$$=\int_a^b (Ax^2+Bx+C)dx = \left(\frac{b-a}{6}\right)\left[p(a)+4p\left(\frac{a+b}{2}\right)+p(b)\right] \quad (11)$$

that requires three consecutive grid points in an even interval [a, b]. The definite integral of the second degree polynomial in equation (11) is simplified by a rule called **prismoidal rule.** This rule is valid for a polynomial p(x) of degree **less than or equal to 3.**

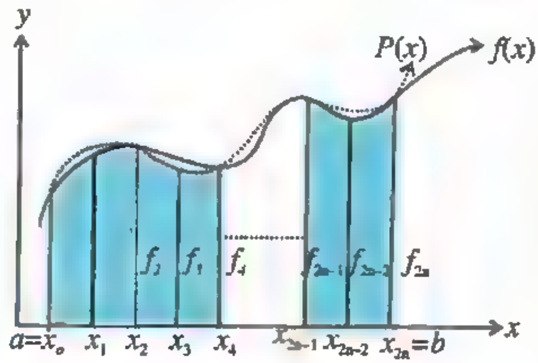


Fig. 11.6

For Simpson's rule, the function f(x) is known at equally even spaced grid points in the interval [a, b]:

$$\left.\begin{array}{l} x:\ a=x_o\ \ x_1\ \ x_2 \ldots x_{2n-2}\ \ x_{2n-1}\ \ x_{2n}=b \\ f(x):\quad f_o\ \ f_1\ \ f_2 \ldots f_{2n-2}\ \ f_{2n-1}\ \ f_{2n} \end{array}\right\},\ with$$

$$x_{2n}=x_o+2n\Delta x,\ n=0,1,2,\ldots.$$

$$b=a+2n\Delta x,\ x_{2n}=b,\ x_0=a$$

$$\Delta x=\frac{b-a}{2n} \qquad (12)$$

In light of even spaced grid points, the definite integral (7) is rearranged as under:

$$I=\int_a^b f(x)dx=\int_{x_o}^{x_2} f(x)dx+\int_{x_2}^{x_4} f(x)dx+\ldots\ldots+\int_{x_{2n-2}}^{x_{2n}} f(x)dx \qquad (13)$$

If $\int_{x_0}^{x_2} f(x)dx$ is the actual area in the initial even subinterval $[x_0, x_2]$, then the parabolic arc in the subinterval $[x_0, x_2]$ represents the approximate area:

$$\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} p(x)dx,\ a=x_0,\ b=x_2,\quad used\ result\ (11)$$

$$=\left(\frac{x_2-x_0}{6}\right)\left[p(x_0)+4p\left(\frac{x_0+x_2}{2}\right)+p(x_2)\right] \qquad (14)$$

If the width of one subinterval is $\Delta x=(b-a)/n$, then the width of two consecutive subintervals is $x_2-x_0=2\Delta x$. The subintervals are of equal width that gives $\frac{x_2+x_0}{2}=x_1$. Using these in equation (14) to obtain

$$\int_{x_0}^{x_2} f(x)dx=\int_{x_0}^{x_2} p(x)dx$$
$$=\frac{2\Delta x}{6}\left[p(x_0)+4p\left(\frac{x_0+x_2}{2}\right)+p(x_2)\right]$$
$$=\frac{\Delta x}{3}\left[p(x_0)+4p(x_1)+p(x_2)\right] \qquad (15)$$

Since the polynomial passes through the three consecutive grid points on the curve, the best approximations would be the function itself: that is $p(x_0)=f(x_0)$, and $p(x_1)=f(x_1)$, and $p(x_2)=f(x_2)$. These are substituted in equation (15) to obtain:

$$\int_{x_0}^{x_2} f(x)dx=\int_{x_0}^{x_2} p(x)dx$$
$$=\frac{\Delta x}{3}\left[p(x_0)+4p(x_1)+p(x_2)\right]$$
$$=\frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] \qquad (16)$$

Thus, the sum S_{2n} of the areas of all n parabolic strips is giving the approximate area under the curve f(x) to actual area represented by definite integral (13):

$$S_{2n}=\frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)\right]+\frac{\Delta x}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right]$$
$$+\ldots+\frac{\Delta x}{3}\left[f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n})\right]$$

$$=\frac{\Delta x}{3}\left[f_0+4(f_1+f_3+\ldots+f_{2n-1})+2(f_2+f_4+\ldots+f_{2n-2})+f_{2n}\right] \qquad (17)$$

Simpson's Rule: If $f(x)$ is continuous on $[a, b]$, then the Simpson's rule is

$$I = \int_a^b f(x) \approx S_{2n} = \frac{\Delta x}{3}\left[f_0 + 4(f_1 + f_3 + ... + f_{2n-1}) + 2(f_2 + f_4 + ... + f_{2n-2}) + f_{2n}\right]$$

The even n[th] subinterval is $x_{2n} = x_0 + 2n\Delta x \Rightarrow b = a + 2n\Delta x$ that gives $\Delta x = ((b-a)/2n$. Moreover, the larger the value for n, the better the approximation.

Example 11.2.2: [Simpson's Rule]: Approximate the definite integral

$$I = \int_{-1}^{2} x^2 dx$$

in n=2 subintervals and then compare your approximate answer with the actual value of the definite integral that must be accurate to 5 decimal places.

Solution: The given interval is $[a, b] = [-1, 2]$ and the width for n=2 subintervals is

$$\Delta x = \frac{b-a}{2n} = \frac{2-(-1)}{4} = \frac{3}{4} = 0.75.$$

The Simpson's rule (17) is used for $\Delta x = 0.75$ and n=2 to obtain

$$\int_{-1}^{2} x^2 dx \approx S_4 = \frac{0.75}{3}\left[f_0 + 4(f_1 + f_3) + 2(f_2) + f_4\right], \quad n = 2 \qquad (18)$$

The function values f_0, f_1, f_2, f_3, f_4 at grid points x_0, x_1, x_2, x_3, x_4 are

$x_0 = a = -1,$ $\quad f_0 = f(-1) = (-1)^2 = 1$

$x_1 = a + 1\Delta x = -1 + \frac{3}{4} = -\frac{1}{4},$ $\quad f_1 = \left(-\frac{1}{4}\right)^2 = \frac{1}{16}0.0625$

$x_2 = a + 2\Delta x = -1 + 2\frac{3}{4} = \frac{1}{2},$ $\quad f_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25$

$x_3 = a + 3\Delta x = -1 + 3\frac{3}{4} = \frac{5}{4},$ $\quad f_3 = \left(\frac{5}{4}\right)^2 = \frac{25}{16} = 0 = 1.5625$

$x_4 = a + 4\Delta x = -1 + 4\frac{3}{4} = 2,$ $\quad f_4 = (2)^2 = 4$

These function values are used in equation (18) to obtain:

$$\int_{-1}^{2} x^2 dx \approx S_4 = \frac{0.75}{4}[f_0 + 4(f_1 + f_3) + 2f_2 + f_4]$$

$$= \frac{0.75}{3}[1 + 4(0.0625 + 1.5625) + 2(0.25) + 4]$$

$$= 0.25(1 + 6.5 + 0.50 + 4) = 3$$

The exact value of the integral is:

$$I = \int_{-1}^{2} x^2 dx = \left|\frac{x^3}{3}\right|_{-1}^{2} = \frac{8}{3} + \frac{1}{3} = 3$$

The error term is therefore:

$$E_2 = I - S_4 = 3 - 3 = 0$$

This develops the idea that Simpson's rule is much more accurate than trapezoidal rule.

Exercise 11.2

1. Use the trapezoidal rule to approximate the value of each definite integral. Round the answer to the nearest hundredth and compare your results with the exact value of the definite integral:

a. $I = \int_{1}^{3} x^2 dx,\ n = 4$

b. $I = \int_{0}^{1} \left(\frac{x^2}{2} + 1\right) dx,\ n = 4$

c. $I = \int_{1}^{3} \frac{dx}{x},\ n = 6$

d. $I = \int_{0}^{2} \sqrt{1 + x^2}\, dx,\ n = 6$

c. $I = \int_{0}^{1} \frac{1}{\sqrt{1 + x^2}} dx,\ n = 5$

2. Use Simpson's rule to approximate the value of each definite integral. Round the answer to the nearest hundredth and compare your results with the exact value of the definite integral:

a. $I = \int_{2}^{4} x^2 dx,\ n = 3$

b. $I = \int_{2}^{3} \left(\frac{x^2}{3} - 1\right) dx,\ n = 4$

c. $I = \int_{1}^{3} \frac{dx}{x},\ n = 3$

d. $I = \int_{0}^{1} e^{2x} dx,\ n = 4$

e. $I = \int_{0}^{1} \frac{1}{2 + x + x^2} dx,\ n = 2$

3. A quarter circle of radius 1 has the equation $y=\sqrt{1-x^2}$ for $0 \le x \le 1$, which means that:

$$\int_1^2 \sqrt{1-x^2}\,dx=\frac{\pi}{4},\ n=4$$

Approximate the definite integral on the left by trapezoidal rule that equals the right side when $\pi=3.1$.

4. A quarter circle of radius 1 has the equation $y=\sqrt{1-x^2}$ for $0 \le x \le 1$, which means that

$$\int_0^1 \sqrt{1-x^2}\,dx=\frac{\pi}{4},\ n=4$$

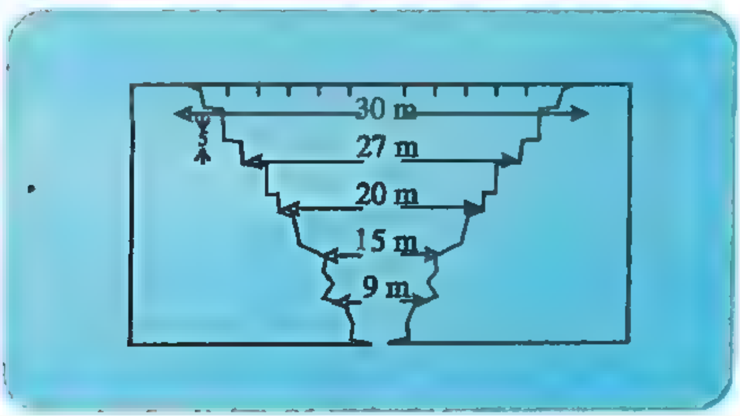
Approximate the definite integral on the left by Simpson's rule that equals the right side when $\pi=3.1$.

5. The width of an irregularly shaped dam is measured at 5-m intervals, with the result indicated in figure given below. Use Simpson's rule to estimate the area of the face of the dam.Hint:

$x : x_0=0 \quad x_1=5 \quad x_2=10 \quad x_3=15 \quad x_4=20$

$f : f_0=9 \quad f_1=15 \quad f_2=20 \quad f_3=27 \quad f_4=30,\ \Delta x=5,\ n=2(\textit{even interval}))$

$$S_4=\frac{\Delta x}{3}[f_o+4(f_1+f_3)+2f_2+f_4]=411.66m^2 \sim 412m^2$$



Glossary

- **Bisection Method:** If f (x) is continuous and $f(a)$ and f (b) have opposite in signs, then the bisection method will be used to approximate the actual root r of the non-linear equation $f(x)=0$ in the interval [a, b]. In bisection method, the approximate root of r will always be the midpoint of the interval [a,b].
- **Regula-Falsi Method:** If f (x) is continuous and $f(a)$ and f (b) have opposite in signs, then the regula-falsi method will be used to approximate the actual root r of the non-linear equation $f(x)=0$ in the interval [a, b]. In Regula-Falsi method, the approximate root of r will not be the midpoint of the interval [a,b] that should be anywhere in the interval [a,b].
- **Newton-Raphson Method:** If f(x) and its derivatives are continuous functions and x_0 is the initial iterate, then the Newton-Raphson method

  $$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)},\quad i=0,1,2,3,\ldots,\ i=1,2,3,\ldots$$

  produces a sequence of successive iterates $\{x_n\}$ that will converge faster to r than either the bisection and regula-falsi methods.
- **Rectangle Rule:** If $f(x)$ is continuous on $[a, b]$, then the rectangle rule

  $$R_n=f(x_1)\Delta x+f(x_2)\Delta x+\ldots+f(x_n)\Delta x$$

  is used to approximate the definite integral $\int_a^b f(x)dx$.
- **Trapezoidal Rule:** If $f(x)$ continuous on $[a, b]$, then the trapezoidal rule is

  $$T_n=\frac{\Delta x}{2}[f_0+2f_1+2f_2+\ldots\ldots+2f_{n-1}+f_n],\quad \Delta x=(b-a)/n$$

  is used to approximate the definite integral $\int_a^b f(x)dx$.
- **Simpson's Rule:** If $f(x)$ is continuous on $[a, b]$, then the Simpson's rule

  $$S_{2n}=\frac{\Delta x}{3}\left[f_0+4(f_1+f_3+\ldots+f_{2n-1})+2(f_2+f_4+\ldots+f_{2n-2})+f_{2n}\right]$$

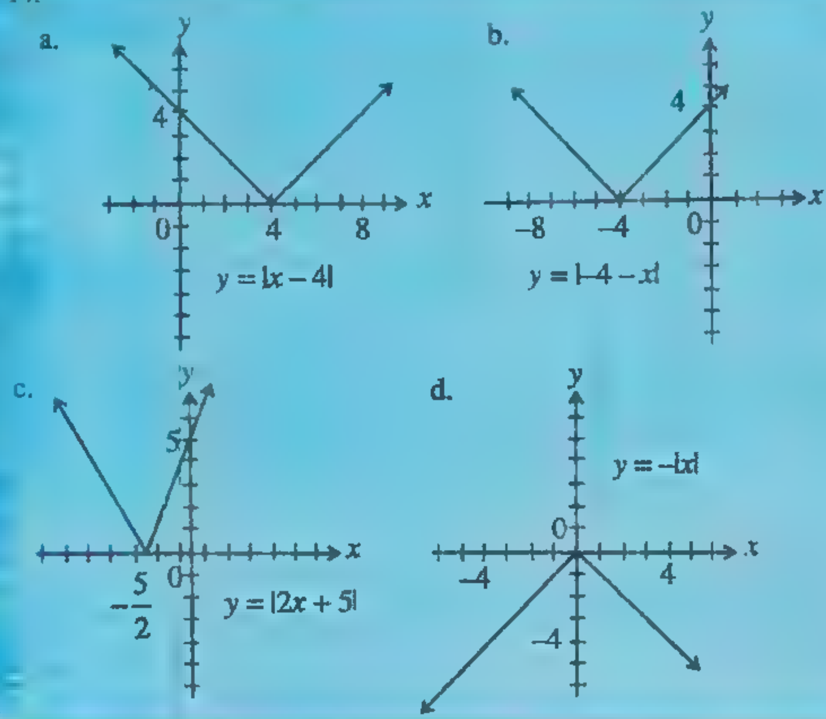
  with $\Delta x=(b-a)/2n$ is used to approximate the definite integral $\int_a^b f(x)dx$.

Answers

Exercise 1.1

1. a. indep: d; dep: P c. indep:F; dep:C
   d. $indep : r, \theta; dep : s$ e. indep:m, a; dep:F
2. a. 43, 15, $3a^2+6ah+3h^2+7a+7h-5$
   b. –7, 2.8, –0.19
   c. 35.69, 103.4, 34.12
3. a. 14.7 in. b. 40.3 m c. 2.856 in.
5. a. $84.4\ m^2$ b. $220\ in^2$ c. $1081\ mm^2$ 1081mm
7. a. –4 b. 5 c. 7
8. a. 2 b. 4 c. 6
9. a. function b. function c. not a function
10. a. not a function b. not a function c. function
11. a. $D=\{x \mid x \in R\},\ R=\{x \mid x \in R\}$
    b. $D=\{t \mid t \in R\},\ R=\{t \mid t \in R\}$
    c. $D=\{r \mid r \geq 0\},\ R=\{f(r) \mid f(r) \geq 0\}$
12. b. $f[g(x)]=\sin(1-x^2),\ g[f(x)]=1-\sin^2 x=\cos^2 x$
    d. $f[g(x)]=u,\ g[f(x)]=u$
    f. $f[g(x)]=\cot x,\ g[f(x)]=\tan(1/x)$
13. a. $f^{-1}(x)=x-5$
    b. $f^{-1}(x)=[x-7]/2$
    c. $f^{-1}(x)=\dfrac{x}{2}+4$
    d. $f^{-1}(x)=2x-4$

14.



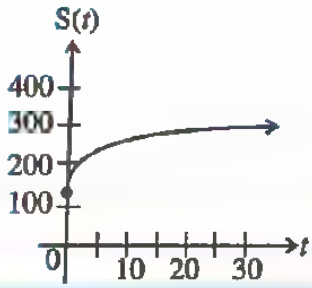
Exercise 1.2

1. a. 4^{6xy} c. e e. $8e^{3.6t}$
2. a. $x=1$ c. $x=-1.6$ e. $x=3$ g. $x=3$ i. $x=-3, 0$
3. a. $27 = 3^3$ c. $10^0 = 1$ e. $8 = 4^{3/2}$
4. a. $\log_7 49 = 2$ c. $\log_4 8 = \frac{3}{2}$ e. $\log_b A = u$
5. a. $x=9$ c. $y=2$ d. $b=10$ e. $y=-2$ f. $b=100$
6. a. $x=2$ c. $x=8$ d. $x=7$

   e. No solution ($x= -11/9$, domain is only positive real numbers)
7. a. \$125,000 b. About \$211,000 c. About \$235,000

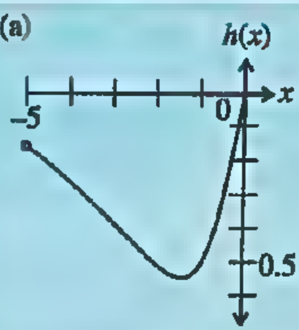
   d. About \$307,000 e. The sales product is growing slowly.

   This is shown in the graph below:

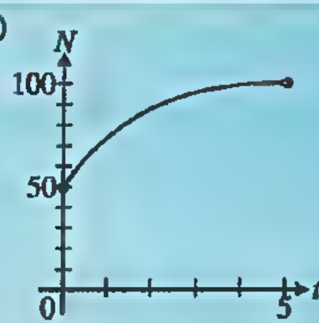


Exercise 1.3

1. (a)

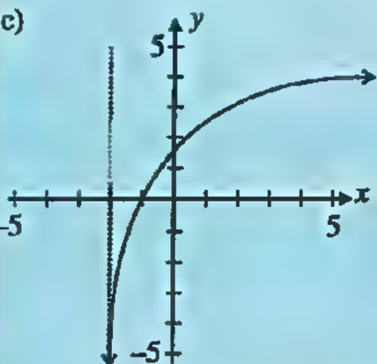


(c)

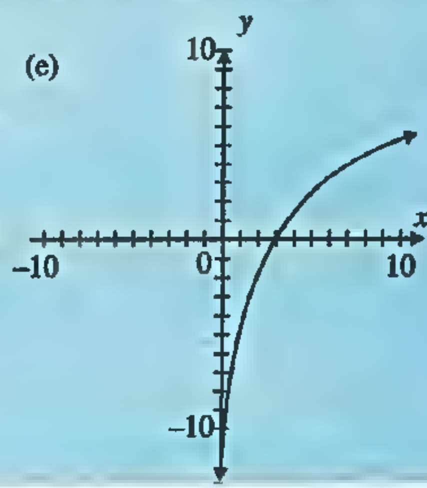


2.

(c)



(e)



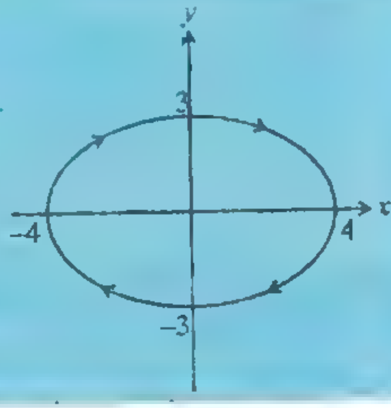
3. a.You should recognize these equations as parametric equations of the straight line parallel to the direction vector u=(–1,2) and passing through the point p(3,0).

b.In order to eliminate the parameter t here, you do not want to solve for the parameter. Instead, look for some relationship between the variables. You should

notice that

$$\left(\frac{x}{4}\right)^2+\left(\frac{y}{-3}\right)^2=\cos^2 t+\sin^2 t=1$$

which is the equation of an ellipse :



c. You should recognize these equations as parametric equations of the straight line parallel to the direction vector u=(2,–3,–4) and passing through the point p(3,5,2).

Exercise 1.4

1. a. –9 b. –8 c. –1/2 e. 9/4 f. ½
   g. –1/9 h. –1 i. 1 j. 1 k. 0 l. 0
2. a. Does not exist. b. –1/9 c. $1/2\sqrt{5}$ d. 1/10
3. a. 5 b. -7
5. a. Continuous for all x. c. Continuous for all x, except x=5.
   e. Continuous for all x, except x=–2 and x=3.
6. a. Yes b. Yes c. Yes d. Yes

Exercise 2.1

1. a. 5 b. 1/3 c. 21 d. 0.25
2. a. 2,2 b. $\Delta x=0.1,\ 2x+\Delta x-6; 0.1$
   c. $\Delta r=0.1,\ \pi(2r+\Delta r); 4.1\pi$ d. $\Delta t=7,\ \dfrac{1}{\sqrt{t+\Delta t}+\sqrt{t}}; 1/7$
3. a. $\Delta t=0.1;\ 14.4$ b. $\Delta t=0.01; 15.84$
4. $\Delta t=2;\ 25;$ the average rate of inflation is Rs.25 per year

Exercise 2.2

1. a. 3 b. 5 c. $2x$ d. $-2x$
   e. $32x-7$ f. $\dfrac{-7}{x^2}$ g. $\dfrac{-3}{(x+3)^2}$ h. $\dfrac{-10}{(2x-4)^2}$
   i. $6x+4$

2. a. 2 b. -1 c. 1/5

Exercise 2.3

1. a. $y' = 9x^2 + 2x - 6$ b. $y' = 120x^3 - 54x^2 + 10$

   c. $y' = 42x^5 - 112x^3 + 6x^2 - 8$

   d. $y' = 50x^4 + 80x^3 - 39x^2 + 16x + 5$

   e. $y' = 3x^{1/2} - (3x^{1/2}/2) - 2$ f. $y' = -12 + 15x^{-1/2}$

2. a. $y' = -7/(x-4)^2$ b. $y' = -6/(3x-5)^2$

   c. $f'(t) = [t^2 - 2t - 1]/(t-1)^2$

   d. $y = [4x^4 - 48x^3 - 2x + 6]/(4x^3 + 1)^2$

   e. $y' = [5\frac{\sqrt{x}}{2} - \frac{3}{\sqrt{x}}]/x$ f. $y' = [x^2 - 4x - 12]/(x-2)^2$

   g. $f(p) = [24p^2 + 32p + 29]/(3p+2)^2$

   h. $g'(x) = [10x^4 + 18x^3 + 6x^2 - 20x - 9]/[(2x+1)^2(5x+2)^2]$

3. a. $C(t) = 1 + 28x^{-1/3}$

   b. About 9190 dollars c. About 9150 dollars d. Yes

4. a. $M'(d) = \dfrac{2000}{(3d+10)^2}$ b. $M'(2) = 125/16,\ M'(5) = 80/25$

6. a. $y = \dfrac{xP}{x-P}$ b. $y' = \dfrac{-P^2}{(x-P)^2}$

7. a. $G'(20) = 1/200$, *go faster* b. $G'(40) = 1/400$, *go slower*

Exercise 2.4

1. a. $-20(7-t)^3$ b. $20(x^3 - 4x + 2)^4(3x^2 - 4)$

   c. $330s(4 - 11s^2)^4$ d. $\dfrac{-33}{5}x^2(4 - x^3)^{10}$

e. $\dfrac{-2t}{\sqrt[3]{(1-3t^2)^2}}$ f. $\dfrac{-21}{(3t+1)^8}$ g. $\dfrac{-16}{(2x-1)^9}$

2. a. $f'(x)=(2x-5)^2(40x-67)$ b. $f'(x)=\dfrac{(x+2)(x-4)}{(x-1)^2}$

c. $f'(x)=\dfrac{-12(2x-5)^3}{(x-4)^5}$ d. $f'(x)=\dfrac{6x^2+11}{(2x^2+11)^{1/2}}$

f. $f'(x)=\dfrac{-9x-74}{(3x-8)^3\sqrt{2x+11}}$ g. $f'(x)=\dfrac{245(3x+8)^6}{(x+9)^8}$

3. a. $y'=\dfrac{3t+4}{2}$ b. $y'=\dfrac{4t^2}{a}$ c. $y'=\dfrac{b(t^2-1)}{2at}$

d. $y'=\dfrac{2t-t^4}{1-2t^3}$

4. a. $F'(t)=150t/(8+t^2)^{3/2}$ (b) *About 3.96, .34, 0.36, 0.09.*

5. $\dfrac{dC}{dt}=\dfrac{dC}{dq}\dfrac{dq}{dt}=4222.80/hr$

Exercise 2.5

1. a. $y'=-x/y$ b. $y'=-y/x$ c. $y'=-[4xy+3y^2]/[2x^2+6xy]$

d. $y'=-[2x+3y]/[3x+2y]$ e. $y'=(x+y)^2/[(x+y)^2+1]$

f. $y'=-y^2/x^2$ g. $y'=-2/3$

2. a. $y'=\dfrac{-2xy}{3(x^2+1)}$ b. $y'=[x^2+4x]/[x+2]^2$, $y'=[2x-y]/[x+2]$

c. $y'=y^2$, $y'=1/[(5-x)^2]$, d. $y'=-3/[(x-1)^2]$, $y'=[1-y]/[x-1]$

3. a. $\dfrac{du}{dv}=-a^2v/b^2u$ b. $\dfrac{dv}{du}=-b^2u/a^2v$

5. $y'=9/7$ 6. $y'=0$

Exercise 2.6

1. a. $y'=1/2\sec\dfrac{2x}{2}$ b. $y'=-\sin(x+\dfrac{\pi}{2})$

c. $y'=\cos(\sin x)\cos x$ d. $y'=\cos^2x-\sin^2x$

e. $y'=\sec^2x$ f. $y'=6\pi x\sin^2(\pi x^2)\cos(\pi x^2)$

2. a. $y' = -6\operatorname{cosec}^2 3x$ b. $y' = \pi \tan \pi x \sec \pi x$

c. $y' = -8\cot 2x \operatorname{cosec} 2x$ d. $y' = 4(x+3)\sec^2(x+3)^2$

e. $y' = \dfrac{-4x}{\sqrt{x^2-1}}\operatorname{cosec}^2\sqrt{x^2-1}$ f. $y' = 6x^2\sec^2 x^3 \tan x^3$

g. $y' = -6\operatorname{cosec}^3(x+2)\cot(x+2)$

h. $y' = \dfrac{2\sec^2 2x + 3\cot 3x(1+\tan 2x)}{\operatorname{cosec} 3x}$

3. a. $y' = \dfrac{-1}{\sqrt{1-(x+4)^2}}$ b. $y' = \dfrac{11}{1+(11x)^2}$

c. $y' = \dfrac{2\sin^{-1}x}{\sqrt{1-x^2}}$ d. $y' = \dfrac{2x^3}{\sqrt{1-4x^2}} + 3x^2\sin^{-1}2x$

e. $y' = \dfrac{-1}{(x+3)\sqrt{(x+3)^2-1}}$ f. $y' = \dfrac{-9(1+\cos^{-1}3x)^2}{\sqrt{1-9x^2}}$

4. a. $P'(t) = \dfrac{5\pi}{26}\sin\dfrac{\pi t}{26}, 0 \le t \le 104$

b. $P'(8) = \$50/week; P'(26) = \$0/week; P'(50) = -\$14/week$

5. a. $V'(t) = \dfrac{0.35\pi}{2}\sin\dfrac{\pi t}{2}, 0 \le t \le 8$

b. $V'(3) = -0.55 liter/sec;\ V'(4) = 0.00 liter/sec; V'(5) = 0.55 liter/sec$

Exercise 2.7

1. a. $y' = \ln x^2 + 2$ b. $y' = [2x+3]/[x^2+3x+2]$

c. $y' = [1-8\ln 5x]/x^9$

d. $y' = x/[x^2+1]$ e. $y' = -2x/[(1+x^2)[\ln(1+x^2)]^2$

f. $y' = -2x/[3(1-x^2)[\ln(1-x^2)]^{2/3}$

2. a. $y' = 5^{x+1}\ln 5$ b. $y' = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$ c. $y' = 2(e^{3t} - e^{-2x})$

d. $y' = 12e^{3x}(e^{3x}-1)$ e. $y' = e^{x\ln x}(1+x+x\ln x)$

f. $y' = (2x-1)(\ln 5)5^{(x^2-x)}$

3. a. $y=\frac{6x\log_e}{3x^2+7}$ b. $y'=\frac{(2x+3).\log e}{x^2+3x+2}$

c. $y'=\frac{(2x-7)\log_e}{2(x^2-7x)}$ d. $y'=\frac{\cot(\log x)}{x}$

e. $y'=\frac{2x\log_{10}e}{\sqrt{1-x^4}\sin^{-1}x^2}$ f. $y'=\sec x$

4 a. $y'=\frac{-1}{x^2-1}$ b. $y'=(\cos x)^{\log x}\left[\frac{\log\cos x}{x}-\tan x\log x\right]$

c. $y'=(1+x^{-1})^x\left[\log(1+x^{-1})-(1+x^{-1})\right]$

d. $y'=\frac{(1-x)^{1/2}(2-x^2)^{2/3}}{(3-x^3)^{3/4}(4-x^4)^{4/5}}\left[-\frac{1}{2(1-x)}-\frac{4x}{3(2-x^2)}+\frac{.9x^2}{4(3-x^3)}+\frac{16x^3}{5(4-x^4)}\right]$

e. $y'=\frac{(2x+15x^2+36)}{3(x^2+3)^{3/2}(x^2+4)^{2/3}}$

f. $y'=(\sin x)(\ln x)(x^x)\left[\cot x+(x\ln x)^{-1}+1+\ln x\right]$

5. a. $y'=(4x+3)\sinh x(2x^2+3x)$ b. $y=\sinh 2xe^{\sinh^2 x}$

c. $y=\tanh x$ d. $y'=-2x[\tanh(x^2+1)sech(x^2+1)-sech^2(x^2+1)]$

e. $y'=-3x^2\coth(x^3+1)cosech(x^3+1)$

g. $y'=\frac{y\cos hx-\cosh y}{x\sinh y-\sinh x}$

6. a. $y'=\sec x$ b. $y'=\sec x$ c. $y'=\sec x\tan x\frac{1}{\sqrt{\sec^2 x-1}}$

d. $y'=\frac{3x}{1-9x^2}+\tanh^{-1}3x$ e. $y'=\cosh^{-1}x$ f. $y'=-\frac{\sqrt{1+y^2}}{\sqrt{x^2-1}\cosh^{-1}x}$

7. 2.27 mm of mercury/yr; 0.81 mm of mercury/yr; 0.41 mm of mercury/yr.

8. $A'(t)=10{,}000(\ln 2)2^{2t}$; $A'(1)=27{,}726$ bacteria/hr (rate of change at the end of the first hour); $A'(5)=7{,}097{,}827$ bacteria/hr (rate of change at the end of the fifth hour).

Exercise 3.1

1. a. $f''(x)=18x$ b. $f'''(x)=-6/x^4$

c. $f''(x)=\frac{4}{x^3}-\frac{18}{x^4}$ d. $s''(t)=\frac{-25}{4}(5t+7)^{-3/2}$

e. $y''=\dfrac{4}{(x-1)^3}$ f. $y''=6x+20$

2. a. $y=6\sec^4x-4\sec^2x$ b. $y=2\cot x\operatorname{cosec}^2x$

c. $y''=3[f(x)]^5-f(x)$ d. $y^{iv}=-\dfrac{(-1)^4 4!}{x^4}$

e. $y''=-\sin x\cos(\sin x)-\cos^2x\sin(\sin x)$

3. a. $y''=\dfrac{-(2y^2+2)}{y^5}$ b. $y''=-\dfrac{r^2}{y^3}$

c. $y''=0$ d. $y''=\dfrac{-b^2(a^2y^2+b^2x^2)}{a^4y^3}$

e. $y''=\dfrac{\tan^2y-\tan^2x}{\tan^3y}$ f. $y''=\dfrac{(1-e^{x+y})(e^x-e^y)}{(e^y+1)^3}$

4. a. $y''=9/32t$ b. $y''=4/3a^2$ c. $y''=-1/a(1-\cos t)^2$

d. $y''=-b/a^2\sin^3 2t$

Exercise 3.2

1. a. $y=(x+5)/4$ b. $y=-2x$ c. $y=x/e$

d. $y=\dfrac{3x}{16}+\dfrac{7}{8}$ e. $y=0.5$ f. $y=-3x+3\pi+1$

g. $y=2x/e$ h. $y=2x+4$

2. a. $y=(-x+2e^2+1)/2e$ b. $y=(x-\pi)/6$ c. $y=(1-x)/2$

d. $y=(-x/12)+1$ e. $y=x+e$ f. $y=-x+(\pi/2)$

g. $y=-x+1$ h. $y=-\dfrac{\sqrt{5}}{2}x+2\sqrt{5}$

3. a. $y-3=\dfrac{2}{3}(x+2)$ b. $y-\pi=(1+\pi)x$ c. $x=1$

5. a. $1-x+x^2-x^3+\ldots$ b. $x^2-\dfrac{1}{3}x^4+\dfrac{2}{45}x^6-\ldots$

c. $1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\dfrac{x^6}{6!}+\ldots$ d. $-4x-8x^2-\dfrac{64}{3}x^3-\ldots$

6. b. 2.7183 7. a. $\pi/4$ b. $\tan^{-1}\sqrt[3]{16}$

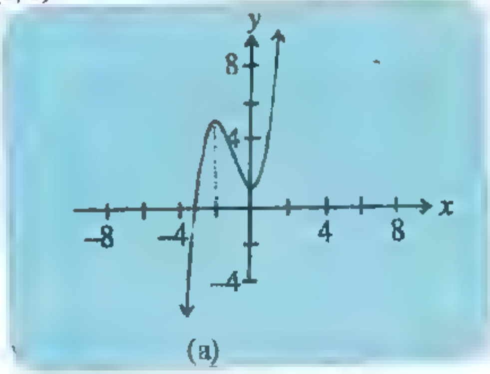
Exercise 3.3

2. a. i. $x = 0, x = -2$.

ii. Increasing on $(-\infty, -2)U(0, +\infty)$; decreasing on $(-2, 0)$

iii. Critical points: (0, 1) is relative minimum; (−2, 5) is relative maximum.

iv. Graph of 2a:

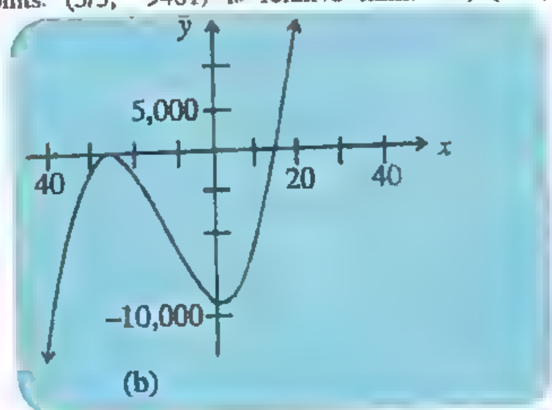


(a)

b. i. $x = 5/3, x = -25$.

ii. Increasing on $(-\infty, -25)U(\frac{5}{3}, +\infty)$; decreasing on (−25,5/3)

iii. Critical points: (5/3, −9481) is relative minimum; (−25, 0). is relative maximum.



(b)

iv. Graph of 2b:

3. a. x=−3, 4 b. x=−3, 5 c. x=0, 1 d. x=0, 1
4. b. Relative minimum at x=1 d. Relative minimum at x=4
5. a. Relative maximum of 1 at x=0, relative minimum of −3 at x=2

   b. Relative maximum of 2 at x=−3, relative minimum of −2 at x=−1
6. Relative maximum at x= 2 , relative minimum at x= 4, neither at x=1 ,−5
7. $p'(x) = 108 - 3x^2 = 0 \Rightarrow x = \pm 6$.

   a. The expenditure on advertising that leads to maximum profit is x=6.

   b. The maximum profit at x=6 is $f(x=6)$=\$512 hundred dollars.

8. $p'(x) = -3x^2 + 18x + 120 = 0 \Rightarrow x = -4, 10.$

The number of hundred thousands of tires is x=1,000,000 tires; the number of hundred thousands of tires x that developed the maximum profit is P(10)=$700=$700,000.

9. $K'(x) = \dfrac{-12x^2 + 108}{(3x^2+27)^2} = 0 \Rightarrow x = \pm 3$

a. The drug concentration is increasing in the interval (0, 3) and is decreasing in the interval $(3, \infty)$. Note: x must be at least 0.

b. The maximum drug concentration time is x=3.

c. The maximum drug concentration at a time x=3 is K (3)=0.22%=0.0022.

10. Hint: $C(x) = [G(x)][32]2.25] = \dfrac{1}{48}\left(\dfrac{300}{x} + 2x\right)(32)(2.25) = 1.5\left(\dfrac{300}{x} + 2x\right)$

$C'(x) = 1.5\left(\dfrac{-300}{x^2} + 2\right) = 0 \Rightarrow x = \pm\sqrt{150}$

Minimum cost at $x = \sqrt{150} = 12.2$ is C (12. 2) = $7.50.

Exercise 4.1

1. a. $t \neq 0,\ t > 0$ b. $t \neq 2,\ t \geq 0.$

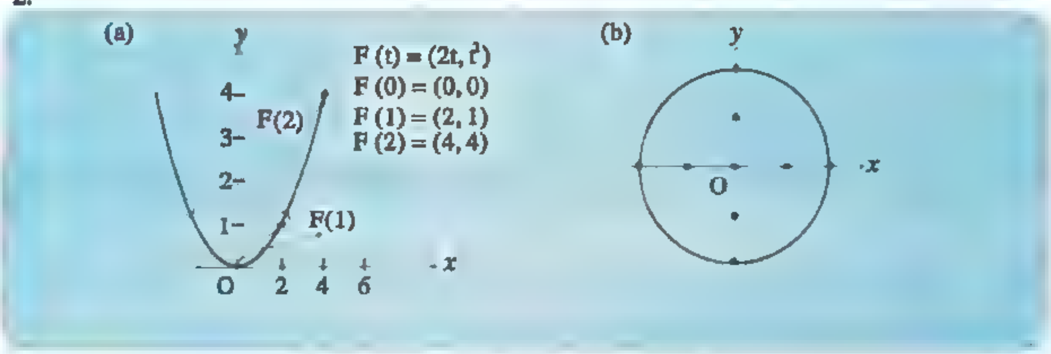
c. $t \neq (2n+1)\dfrac{\pi}{2},\ \ n = \pm 1, \pm 2 \ldots$ since $tant = \dfrac{sint}{cost}$ is undefined for

$t = (2n+1)\dfrac{\pi}{2}, n = 0, \pm 1, \pm 2 \ldots,\ t \neq \dfrac{n\pi}{2}$, n is odd.

d. $t \neq n\pi,\ n = 0, 1, 2, \ldots$ e. $t \neq 0,\ t \geq 10$ or t>0.

f. $t \neq 0,\ t > 0$ g. $t \neq -2,\ t \leq 0$

2.



3. a. $(7t-3)i-10j+(2t^2-\frac{3}{t})k$ b. $3t-2t^2$ c. $(1-t)\sin t$

d. $-t^2e^ti+(t^2\sin t)j+(2te^t+5\sin t)k$

e. $(4te^t+t-t^2+10\sin t)i+(2t^2e^t+1)k$

4. a. $3i+e^2j$ c. $3i-(1/3)j+2k$ e. $i+ek$ g. $2i-3j+ek$

5. a. All values of t. b. All values of t except $t=0$

c. All values of t except $t=0, t=-1$

e. All values of t except $t=0$ f. All values of t except $t=0, t=-1$

Exercise 4.2

1. a. $F'(t)=i+2tj+(1+3t^2)k$

2. a. $F'(t)=2ti-t^{-2}j+2e^{2t}k,\ F''(t)=2i+2t^{-3}j+4e^{2t}k$

c. $F'(s)=\cos si-\sin sj+2sk,\ F''(s)=-\sin si-\cos sj+2k$

3. a. $f'(x)=-9x^2-2x$ c. $g'(x)=\dfrac{4x}{\sqrt{1+4x^2}}$

4.

a. $V(t)=i+2tj+2k;\ V(1)=i+2j+2k,\ |V(1)|=3$

in the direction of $\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k$, A(t)=2j; A(1)=2j

c. $V(t)=-\sin ti+\cos tj+3k,\ V(\frac{\pi}{4})=\frac{\sqrt{2}}{2}i+\frac{\sqrt{2}}{2}j+3k$

$|V(\pi/4)|=\sqrt{10}$ *in the direction of* $\frac{1}{2\sqrt{5}}i+\frac{1}{2\sqrt{5}}j+\frac{3}{\sqrt{10}}k$

$A(t)=-\cos t\,i-\sin t\,j,\ A(\pi/4)=-\frac{\sqrt{2}}{2}i-\frac{\sqrt{2}}{2}j$

5. a. $i+2j-3k$ c. $2\sqrt{14}$

6. a. $[6t+2\cos(2-t)]i+9\sin 3tj-(3t^{-2}-4e^{2t})k$

Exercise 5.1

1. a. $\frac{x^5}{5}+3\frac{x^4}{4}-7x+C$ b. $-\frac{1}{2x^2}+C$

c. $\frac{2}{5}x^5+3x^{\frac{1}{3}}+\frac{3}{2}x^{-\frac{2}{3}}+C$ d. $\frac{t^4}{4}+\frac{3t^5}{5}+C$ e. $\frac{t^5}{5}-\frac{2}{3}t^3+t+C$

f. $\frac{2x^3-1}{2x^2}+C$ g. $\frac{2}{7}z^{\frac{7}{2}}+C$

2. a. $\frac{(3x+4)^9}{27}+C$ b. $\frac{(x^3-4)^2}{2}+C$ c. $\frac{(x^3+7x)^9}{9}+C$

d. $\frac{-1}{3(x^3-7x)^3}+C$ e. $\frac{2x^{\frac{3}{2}}}{3}+\frac{6x^{\frac{5}{2}}}{5}+C$ f. $\frac{-1}{2(x^2+2x+2)}+C$

3. a. $e^{6t}+C$ b. $\frac{e^{5x^2+1}}{10}+C$ c. $\frac{1}{3}e^{x^3-6x+4}+C$

d. $\frac{8^{(7-3x^2)}}{\ln 8}+C$

4. $dy/dx=4x^3+6x^2,\ y=x^4+2x^3-3$

5. $dy/dx=x(2x^2-1)^2,\ y(x)=\frac{1}{12}(2x^2-1)^3-406.42$

7. a. $W(h)=0.0005h^3$ b. $W(70)=171.5lb$

8. N(9)=19,400

Exercise 5.2

1. a. $\frac{\sin^5 x}{5}+C$ b. $\frac{[\ln(\sin x)]^2}{2}+C$ c. $-\cos e^x+C$

d. $\frac{1}{2}\sin(t+3)^2+C$

2. a. $\frac{1}{2}\sec 2x+c$ b. $\tan 4x+C$ c. $\frac{1}{2}\tan^2 x+C$

d. $\frac{1}{3}[\ln|\sec 3x|+\ln|\sec 3x+\tan 3x|]+C$

e. $-\frac{1}{3}\cos^3 x+C$ f. $2\ln|\sin\sqrt{x}|+C$

g. $-\ln|\sin x+\cos x|+C$ h. $-\frac{1}{2}\ln(3+2\cos x)+C$

3. a. $\frac{1}{4}\tan^{-1}\frac{x}{4}+C$ b. $-\tan^{-1}(\cos x)+C$

c. $\frac{\sqrt{2}}{2}\sin^{-1}\left(\frac{x\sqrt{10}}{5}\right)+C$ d. $\frac{1}{2}\sec^{-1}\left(\frac{e^x}{2}\right)+C$

e. $\ln(x^2+4x+5)+\tan^{-1}(x+2)+C$

f. $-\sqrt{4-2x-x^2}+\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right)+C$ g. $\frac{1}{\sqrt{5}}\operatorname{cosec}^{-1}\left(\frac{\sqrt{7}x}{\sqrt{5}}\right)+C$

Exercise 5.3

1. a. $ln\sqrt[3]{\frac{x-3}{c}}+c$ b. $3x-ln+c$ c. $\frac{1}{x}+3lx+\frac{2}{x+1}+lncx+D+C$

d. $\frac{1}{3}\ ln\ (x-1)-\frac{1}{6}\ ln\ (x^2+x+1)-(x-1)-\frac{1}{3}tan\frac{(2x+1)}{\sqrt{3}}+C$

e. $\frac{(x+1)^2}{2}-2\ lnx+\frac{2}{x}\quad 2ln\ (x-1)+C$

2. a. $\frac{1}{2}\left|\frac{x-1}{x+1}\right|+C$ b. $\ln|(x+3)(x-1)^2|+C$

c. $\ln\left|\frac{(2x+1)^{5/6}}{(x-1)^{4/3}}\right|+C$ d. $\ln\left|\frac{(x-5)^3}{(x+3)}\right|+C$

e. $\frac{2}{x+1}-\frac{1}{2(x+1)^2}+\ln|x+1|+C$

f. $\frac{1}{2}\tan^{-1}x+\ln\left|\frac{(x^2+1)^{1/4}}{(x+1)^{1/2}}\right|+C$

h. $x+\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|-\tan^{-1}x+C$ g. $\frac{3}{2}\tan^{-1}x+\frac{1}{4}\sin(2\tan^{-1}x)+C$

3. a. $\frac{e^{2x}}{4}(2x-1)+C$ b. $\sin x-x\cos x+C$

c. $x\tan^{-1}x-\frac{1}{2}\ln(1+x^2)+C$ d. $x\sin^{-1}x+\sqrt{1-x^2}+C$

e. $\frac{x^2(x-3)^{12}}{12}-\frac{x(x-3)^{13}}{78}+\frac{(x-3)^{14}}{1092}+C$ f. $\frac{e^x}{2}(\sin x+\cos x)+C$

g. $\frac{x^3}{3}+\frac{x}{2}-\frac{1}{4}\sin 2x+2\ (sim\ (-xcosn)+C$ h. $\frac{2}{15}(1-e^x)^{3/2}(3e^x+2)+C$

i. $\frac{-x\cos 2x}{4}+\frac{\sin 2x}{4}+C$ j. $\frac{x^3}{3}\ln x-\frac{x^3}{9}+C$

4. b. $\frac{1}{2}\cos^2 x-\cos^2 x\ln(\cos x)+C$
5. Use partial fractions; $R(1)=4.522$ ml; Fourth hour = R(4) –R(3)=1.899 ml
6. Use partial fractions; 45 thousands
7. 110 ft.

Exercise 5.4

1.	a. 8	b. 7	c. 6.5	
2.	a. 7	b. 5.75	c. 5.1875	
3.	a. 13.5	b. –3	c. 2.67	d. 2.33

Exercise 5.5

1. a. 5 b. 8 c. –2 d. 14 e. 15625
   f. 18.667 g. 9.333 h. 25.918
2. a. 7.819 b. 0.928 c. -0.203 d. 0
3. a. $-2+\sqrt{2}$ b. -0.28 c.1 d. 0.241 e. $\pi/4$
   f. $\pi/12$ h. $\sin^{-1}\frac{9}{2a}$
4. a. -1.609 b. 28.2 d. 0.458 e. 3.16
   g. $15/56$ h. –4.1548
5. a. $Area=\int_0^2(4-x^2)dx-\int_2^3(4-x^2)dx=23/3,$ *square units*

   b. $Area=\int_0^2(x^2-5x+6)dx-\int_2^3(x^2-5x+6)dx=29/6,$ *square units*

   c. $Area=\int_0^2(x^2-6x+8)dx-\int_2^4(x^2-6x+8)dx=8,$ *square units*

   d. $Area=\int_1^3(5x-x^2)dx=34/3=11.34$
6. a. 23/6 square units b. 8/3 c. 4.01 square units
7. a. About 414 barrels (one day means 24 hours) b. About 191 barrels
   c. Decreasing to 0

Exercise 6.1

1. a. 2 b. 11 c. 11 d. 3
2. $|AB|=|BC|=\sqrt{13}$, $|AC|=2\sqrt{13}$
3. Sides: $|AB|=|BC|=|CD|=|AD|=\sqrt{18}$, Diagonals: $||AC||=|BD||=6$
5. Sides: $|AB|=|CD|=\sqrt{180}$, $|BC|=|AD|=\sqrt{500}$

   Diagonals: $|AC|=|BD|=\sqrt{680}$
6. $|AB|=\sqrt{5}$, $|BC|=\sqrt{5}$, triangle ABC is isosceles.
8. a. (–1,6) b. (0,0) c. (a,a)
9. a. (11/6,17/6) b. (–24,–2) c. (5,–20) d. (–32/9,41/9).
10. a. Q(5,1) b. P(–8,–12) c. Q(–2,–1)
11. a. 3:4 b. 1:2 c. 1:2 d. 1:3
12. a. G(7/3,7/3) b. G(10/3,10/3) c. G(1,4/3) d. G(2,3)

Exercise 6.2

1. a. y=x+5 b. y=–3x+6 c. y=2x+1
   d. y=(–1/2)x+1 e. y=x f. y=(–3/4)x–3
   g. x=–7 h. y=3
2. e. d, b, a, c
3. a. 1/4 b. –1 c. –3/2
   d. –4/7 e. 2 f. –1/6
4. a. –3, 6 b. 3, 9 c. –2, 2
   d. 4, –8 e. –2, 10 f. 2, –10
5. a. $0.5x+0.87y-4=0$ b. $y-8=0$
   c. $0.87x+0.5y-5=0$ d. $0.71x+0.71y-2=0$
6. a. $3x+2y-12=0$ b. $2x+y-8=0$
   c. $2x+y-10=0$ d. $5x-y+5=0$
7. a. $y=3x$ b. $5x-y-5=0$
   c. $y-x=0$ d. $x+2y-5=0$
8. a. $4x-y-2=0$ b. $x+2y+5=0$
   c. $3x+y+4=0$ d. $5x-y-43=0$
9. a. $2x-y+2=0$ b. $8x-y+4=0$
   c. $x-2y-8=0$ d. $6x+2y-1=0$

Exercise 6.3

1. a. Below the line; on the opposite sides of the line
   b. Above the line; on the same sides of the line
   c. Below the line; on the opposite sides of the line
2. a. On the opposite sides   b. On the same sides   c. On the opposite sides
3. a. 6   b. $6/\sqrt{13}$ c. 1   d. 4   e. 0
4. a. 45^0 b. $\tan^{-1}\frac{1}{7}$, $\tan^{-1}\frac{1}{7}$ c. $180^0 - \tan^{-1}\frac{11}{23}$, $\tan^{-1}\frac{11}{23}$
   d. $\tan^{-1} 2$
5. a. $\tan^{-1}(\infty) = 90^0$ b. $180^0 - \tan^{-1}\frac{7}{6}$, $\tan^{-1}\frac{7}{6}$
6. a. $\theta = \tan^{-1}(-1)$ b. $\theta = \tan^{-1}(13)$ c. $\theta = \tan^{-1}\frac{-7}{8}$
7. a. $\tan^{-1}\frac{27}{11}, \tan^{-1}\frac{27}{14}, \tan^{-1}\frac{27}{23}$ b. $\tan^{-1}\frac{9}{2}, \tan^{-1}\frac{9}{83}, 180^0 - \tan^{-1}(9)$
   c. $\tan^{-1}\frac{5}{6}, 90^0, \tan^{-1}\frac{6}{5}$ d. $\tan^{-1}(3), \tan^{-1}\frac{3}{4}, \tan^{-1}(3)$
8. a. $119x+102y=125$ b. $23x+23y=11$
   c. $3x+4y=5a$ d. $x-3y=0$
9. a. $x+y=0$ b. $4x-5y-6=0$

Exercise 6.4

1. a. (1,2)   b. (14,–20)   c. (8/3,6)   d. (13,15)
2. a. concurrent; (3,1)   b. concurrent; (–1,2)   c. not concurrent
   d. concurrent; (23,–14)
4. a. 6   b. 15   c. 5/2
5. 2ac
7. a. $x+y=0$, $3x-5y=0$ b. $x-y=0$, $4x-5y=0$
   c. $\left(\frac{1+\sqrt{-3}}{2}\right)x + y = 0$, $\left(\frac{1-\sqrt{-3}}{2}\right)x + y = 0$,   d. $x-y=0$, $x-6y=0$
8. a. $\theta = \tan^{-1}(-4)$ b. $\theta = \tan^{-1}\frac{1}{9}$
   c. $\theta = \tan^{-1}\frac{\sqrt{-3}}{2}$ d. $\theta = \tan^{-1}\frac{5}{7}$

9. a. $\left(\frac{5-\sqrt{29}}{2}\right)x - y = 0,\ 1\left(\frac{5+\sqrt{29}}{2}\right)x - y = 0,$; perpendicular.

b. 2x+y=0, x−y=0; neither.

c. $\left(\frac{1+j\sqrt{3}}{6}\right)x + y = 0,\ 2\left(\frac{1-j\sqrt{3}}{6}\right)x + y = 0,$; parallel.

10. a. $2x^2+7xy+3y^2=0$ b. $x^2-2\tan\theta xy-y^2=0$

c. $bx^2-2hxy+ay^2=0$

Exercise 7.1

1. a. $x^2+y^2=16$ b. $(x-3)^2+(y-2)^2=1$

c. $(x+4)^2+(y+3)^2=16$ d. $(x+a)^2+(y+b)^2=(a+b)^2$

2. a. $x^2+y^2=25$ c. $(x-6)^2+(y+6)^2=72$

e. $(x+9)^2+(y+6)^2=317$ g. $(x+5)^2+(y-4)^2=16$

3. a. (4, 3), 4 b. $(-2, 3/2)$, $2\sqrt{2}$ c. (− 2, 3), 0 d. (− 1/2, − 4), 137/4

4. a. (4, 2), r = 2 c. $(0, 1/2)$, $r=3/2$ e. $(-1, 1)$, $r=\sqrt{2}$

g. No, coefficients of x^2 and y^2 are not equal. i. No, $r^2 \geq 0$

5. a. $x^2+y^2-4x-21=0$ b. $x^2+y^2+4x+6y-72=0$

c. $x^2+y^2-22x-4y+25=0$ d. $x^2+y^2+x-5y-2=0$

6. a. $x^2+y^2-2x+2y-48=0$ b. $x^2+y^2-6x-8y+15=0$

c. $3x^2+3y^2+10x-18y-61=0$ d. $x^2+y^2-2x-9=0$

7. a. $x^2+y^2+\frac{12x}{5}-3y=0$ b. $x^2+y^2-3x+y=0$

8. The center of the required concentric circle is (-g, -f)=(3/2,2) and the radius which is the distance from the center (3/2,2) to the point (-3,0) on the circle:

$$r=d=\sqrt{\left(\frac{3}{2}+3\right)^2+(2-0)^2}=\sqrt{\frac{97}{4}}.$$

The required concentric circle is:

$$\left(x-\frac{3}{2}\right)^2+(y-2)^2=\frac{97}{4} \Rightarrow x^2+y^2-3x-4y-18=0$$

9. a. The center of the required concentric circle is (-g, -f)=(-4,7/4) and the radius which is the perpendicular distance from the center (-4,7/4) to the tangent line x=0 on the circle:

$$r=d=\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}=\frac{1(-4)}{\sqrt{1^2}}=-4$$. The required concentric circle is:

$$(x+4)^2+\left(y-\frac{7}{4}\right)^2=16$$

b. $(x-4)^2+(y-0)^2=20$ c. $(x+3)^2+(y-5)^2=25$

10. a. The circle equation $x^2+y^2+2gx+2fy+c=0$ that passes through the origin is giving a circle equation $x^2+y^2+2gx+2fy=0,\ c=0$. Intercepts means the x-intercept and y-intercept that are x=3, y=0 and y=4, x=0. These intercepts develop the two points (3,0) and (0,4). The circle equation $x^2+y^2+2gx+2fy=0$ through these two points (3,0) and (0,4) give the values of g=-3/2 and f=-2. The circle equation $x^2+y^2+2gx+2fy=0$ through these values of g=-3/2 and f =-2 is giving the required circle equation of the form $x^2+y^2-3x-4y=0$.

b. $x^2+y^2-2x-4y=0$

Exercise 7.2

1. a. $x+2y=5;\ y=2x$

   b. $4x+5y=11;\ 5x-4y+17=0$

2. a. $x+y=2\sqrt{2};\ x-y=0$

3. a. $9(1+1)-n^2=0 \Rightarrow 18-n^2=0$

   b. $81(4+4)-n^2=0 \Rightarrow 81(8)-n^2 \Rightarrow 648-n^2=0$

4. a. $n^2=18 \Rightarrow n=\sqrt{18}=3\sqrt{2}$

   b. $n^2=648 \Rightarrow n=\pm\sqrt{648}=\pm18\sqrt{2}$

5. a. $c^2=a^2(1+m^2)\Rightarrow c=\pm a\sqrt{1+m^2}$ b. $c^2=9(1+1)\Rightarrow c=\sqrt{18}=3\sqrt{2}$

6. $(c-f^2)l^2+2fglm+(c-g^2)m^2-2n(gl+fm)+n^2=0$

7. a. $n=7,\ -43$ b. $n=4.52,\ 13.52$

8. a. $(3x+4y+3)^2=25(x^2+y^2-9)$

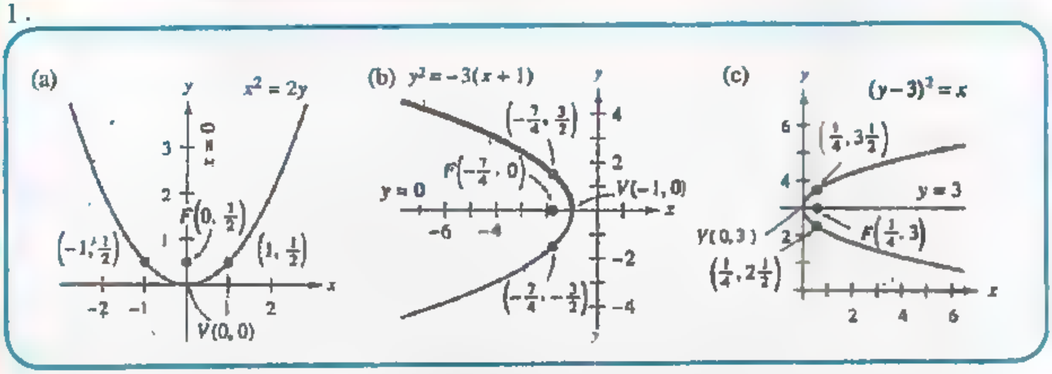
11. a. $(8,2),(3,7)$ b. (1,0), (1,2)

12. $y=mx\pm c=\tan 45°\ x\pm c=x\pm 2,\ \tan 45°=1$

13. a. $y=3, y=\frac{24}{7}x\pm\frac{75}{7}$,(choose negative sign)

Exercise 8.1

1.



2. a. $(x-2)^2=8(y+1)$ b. $x^2+2x-3y+1=0$

3. a. $12y=x^2$ b. $y^2=16x$ c. $y^2=12x3$

e. $(x-5)^2=(64/3)(y-1)$

4. $(x-4)^2+(y-3)^2=(x+2)^2+(y-1)^2 \Rightarrow 3x+y=5$

6. a. (1,3), (4,6) b. (–3,–1) (–4,–2)
c. (2,0), (2,0) d. (2,0), (2,0)

7. a. c=p/m=9/4, m=1 b. c=2
c. c=–2 d. c=-1/6

8. a. x–y+2=0, x+y–3=0 b. y=x+3, x+y–9=0
c. 4x+2y–3=0, 4x–8y+27=0 d. 4x+3y–1=0, 8x–9y–7=0

9. a. 2x+y+1=0, (1/2,–2) b. y=(–1/4)x–5, (20,–10)

10. a. $y=mx+c=\tan 45^0x+c=x+\frac{p}{m}=x+\frac{1}{4}$, $p=1/4, m=\tan 45^0=1$

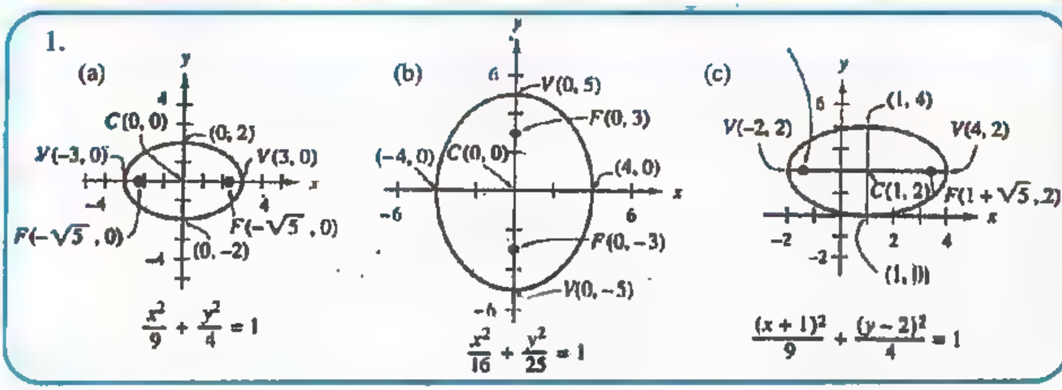
b. $y=mx+c=\tan 60^0x+c=\sqrt{3}x+\frac{p}{m}=\sqrt{3}x+\frac{1}{2\sqrt{3}}$,

$p=\frac{1}{2}, m=\tan 60^0=\sqrt{3}$

12. $x^2=-6(y+3)$

Exercise 8.2:

1.



2. a $\frac{x^2}{36}+\frac{y^2}{9}=1$ b. $\frac{(x-2)^2}{9}+\frac{y^2}{25}=1$

c. $\frac{(x-6)^2}{49}+\frac{(y+2)^2}{16}=1$

3. a. $\frac{(x+3)^2}{4}+\frac{(y-2)^2}{1}=1$ b. $\frac{(x-8)^2}{16}+\frac{(y-2)^2}{4}=1$

c. $\frac{(x-1)^2}{25}+\frac{(y-3)^2}{9}=1$ d. $\frac{x^4}{4}+\frac{(y-5)^2}{9}=1$

4. a. $e=c/a=3/5=0.6$

b. $\frac{x^2}{25}+\frac{y^2}{16}=1, a=5, c=3, b^2=a^2-c^2=16$

c. $e=c/a=\sqrt{12}/4=0.87$

5. a. $c=\pm\sqrt{5}$ b. $c=\pm 4$ c. $c=\pm 6$

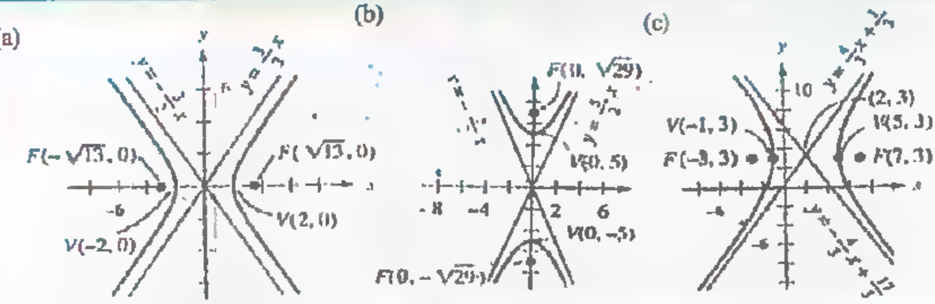
6. a. $2x+3y=8,\ \ 3(x-\frac{7}{4})=2(y-\frac{3}{2})$ b. y=2, x=0

c. $\sqrt{3}x+2y=4,\ \ 2(x-\sqrt{3})=\sqrt{3}(y-\frac{1}{2})$

7. a. $8x-9y-1=0$ b. $2x+7y-14=0$

Exercise 8.3

1. (a) (b) (c)



2. a. $\frac{x^2}{16}-\frac{y^2}{4}=1$ b. $\frac{y^2}{25}-\frac{x^2}{16}=1$

c. $\frac{(x-4)^2}{9}-\frac{4(y-3)^2}{9}=1$ d. $\frac{(y+2)^2}{9}-\frac{(x+5)^2}{16}=1$

3. a. $\frac{y^2}{4}-\frac{x^2}{5}=1$ b. $x^2-y^2=1$ c. $\frac{x^2}{25}-\frac{y^2}{24}=1$

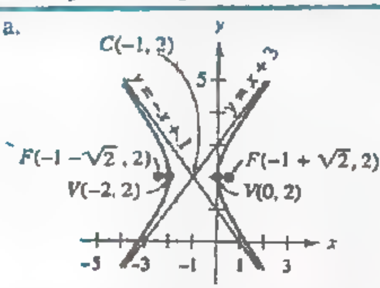
d. $\frac{x^2}{9}-\frac{y^2}{81}=1$ e. $9x^2-y^2=9$ f. $y^2-9x^2=9$

g. $16x^2-9y^2=144$ h. $\frac{x^2}{81}-\frac{y^2}{81}=1$

4. a. $\frac{x^2}{16}-\frac{y^2}{9}=1$ b. $\frac{y^2}{25}-\frac{x^2}{144}=1$

5. a. $\frac{(x+1)^2}{9}-\frac{(y-2)^2}{3}=1$

6. a.



$(x+1)^2-(y-2)^2=1$

7. a. (−1,−4), (4,1) b. (1,0), $(2,\sqrt{3})$

c. (0,1), (−8/3,5/3) d. (0,3), (−8/3,−5)

8. a. $c=\pm 2\sqrt{3}$ b. $c=\pm\sqrt{7}$ c. $c=\pm\sqrt{3}$

9. a. $y=\frac{-\sqrt{13}}{2}x-2,\ y=\frac{2}{\sqrt{13}}x+\frac{13}{2}$ b $y=\frac{3}{5}x+\frac{9}{5},\ y=\frac{-5}{3}x+\frac{125}{9}$

Exercise 8.4

1. a. 2X-Y=0 b. $X^2+Y^2=4$ c. $X^2+2Y^2+16=0$
2. a. $2XY+a^2=0$ b. $X^2=-4pY$ c. $3X^2-Y^2-1=0$
3. a. $3X^2+2Y^2=5$ b. $X^2-4Y^2+1=0$
4. a. $\theta\approx 37^0$ b. $\theta=-30^0$

Exercise 9.1

1. a. order = degree=1, linear b. order = 2, degree = 1, linear
   c. order = 3, degree = 1, non-linear
2. a. y is a solution, since the substitution of y and its derivative y' reduce the differential equation into an identity.
   b. y is a solution.
   c. y is a solution
3. a. Non-linear in x and y b. Linear in x and y c. Linear in x and y
4. a. $xy=2$ b. $y=x-x\ln x+1$ c. $\sin(xy)=0.7071$
   d. $\frac{y^2}{x}=\frac{x^2}{2}+\frac{1}{2}$
5. a. $y=\sin x+1$, obtained by direct integration.
   b. $y=\frac{x^3}{3}+1$, obtained by shifting x on one side and y on the other side.
   c. $y=\frac{1}{8-x^2}$, obtained by shifting x on one side and y on the other side.
   d. $y=1/(1+e^x)$, obtained by shifting x on one side and y on the other side through partial fractions.
   e. y=-1, obtained by shifting x on one side and y on the other side through substitution $y^2=z$.
   f. $y=-2xe^{-x}-2e^{-x}+44$, obtained by shifting x on one side and y on the other side.

Exercise 9.2

1. a. y=cx b. $y=2-ce^{\frac{-x^2}{2}}$ c. $y=\pm\sin(x+c)$
   d. $y=\frac{-1}{e^{-x}+c}$ e. $y=\sin(\sin^{-1}x+c)$

f. $y=\sin^{-1}\left(\frac{\sin 2x}{4}-\frac{x}{2}+c\right)$

2. a. $y=\tan(x+c)-x$ b. $\sin(x+y)=ce^{x}$ c. $e^{y}=ce^{x}-x$

d. $y^2=1-ce^{-x^2}$ e. $e^{\left(\frac{y}{x}\right)^3}=cx^3$ f. $y^2-6y=\frac{x^4}{2}+2x+c$

3. a.. $\tan^{-1}\frac{y}{x}=\ln\left[cx\left(1+\frac{y^2}{x^2}\right)^{\frac{1}{2}}\right]$ b. $x=ce^{\frac{x}{y}}$

c. $x^3=c(x^2+y^2)$ d. $e^{\sin^{-1}(y/x)}=cx$

e. $y^2=c(x^2+2xy-y^2\ln y^2)$

4. a. $y=\frac{1}{4}x^2-1$ b. $y^2=(x^2\ln x)/(1+\ln x)$

5. $y=\frac{x-13}{x-1}$ 6. $\left(\frac{2y^2}{x^2}+1\right)^{\frac{1}{4}}=9^{\frac{1}{4}}x$

8. $\frac{dx}{dt}=\frac{x}{2}-3t,\ x(t)=6t+12+ce^{\frac{t}{2}},\ x(t)=6t+12-28e^{\left(\frac{t}{2}-1\right)}$

9. General consumption of oil is $x(t)=30e^{0.04t}+c$. At time $t=0$ (1990), the oil consumption was $x=0$, that gives $c=-30$. The actual oil consumption at time t is $x(t)=30e^{0.04t}-30$.

10. General number of infected peoples is $I(t)=50\ln(t^2+1)+c$. At time t = 0, the number of infected peoples was I = 0, that gives c = 0.

The actual number of infected peoples at a time t is

$I(t)=50\ln(t^2+1)$

The total numbers of infected people over the first four months of the disease are the following

$I(1)=34.66,\quad I(2)=80.17,\quad I(3)=115.13,\ I(4)=141.66\ peoples$

11. General reaction of drug is $R(x)=c-2e^{-x}(x^2+2x+2)$. At time x = 0, the drug reaction was R = 0, that gives c = 4.

The actual drug reaction at a time x is

$R(x)=4-2e^{-x}(x^2+2x+2)$

The total drug reaction from x = 1 (hour) to x = 6 (hours) are following:

$R(1) = 0.321 = 0.321(100\%) = 32.1\%,$

$R(2) = 1.293 = 1.293(100\%) = 129.3\%$

$R(3) = 2.307 = 2.307(100\%) = 230.7\%$..........

12. $s(x) = 0.19x^2 + .04x + c,\ c = 0.25, s(6) = 7.33\ millions$

13. a. $x^2 + 3y^2 = k$ b. $y^2 - x^2 = k$ c. $\frac{y^2}{2} = -x + \ln k(x+1)$

d. $yx = k$ e. $\frac{y^2}{2}\ln k(\cos 2x)^{\frac{1}{4}}$ f. $e^x \sin y = k$

g. $y = ke^{-2x}$ h. $2y = -\ln x + k$ i. $e^{-x}\cos y = k$

j. $\sin x \sinh y = k$ k. $\sin y = ke^{-x}$

Exercise 10.1

1. a. 0 b. $1 - e^2$ c. $e^{-2} + 1$ d. $2x^3e^{2x} + 3x^2e^{2x} + 2x$

e. $e^2 + 2y$ f. $4z(z^2 - 2)$

2. a. Domain: $x > y$; range: $f(x, y) > 0$

b. Domain $\frac{y}{x} \geq 0,\ x \neq 0$; range: $f(x, y) \geq 0$

c. Domain: $y \neq 2$; range: $f(x, y) > 0$

d. Domain: $x^2 + y^2 < 9$; range: $f(x, y) > 0$

3. a. $f_x = 2x\cos x^2 \cos y,\ f_y = -\sin x^2 \sin y$

b. $f_x = \frac{3x}{\sqrt{3x^2 + y^4}},\ f_y = \frac{2y^3}{\sqrt{3x^2 + y^4}}$

c. $f_x = y^3 \tan^{-1} y,\ f_y = x\left(\frac{y^3}{1+y^2} + 3y^2 \tan^{-1} y\right)$

d. $f_x = 3x^2 + 2xy + y^2,\ f_y = x^2 + 2xy + 3y^2$

e. $f_x = \frac{y}{\sqrt{1 - x^2y^2}},\ f_y = \frac{x}{\sqrt{1 - x^2y^2}}$

f. $f_x = x(x+2)e^{x+y}\cos y =,\ f_y = x^2e^{x+y}(\cos y - \sin y)$

4. $f_x = 0.7x^{-0.3}y^{0.3},\ f_y = 0.3x^{0.7}y^{-0.7}$

Exercise 10.2

1. a. Not homogeneous b. Homogeneous, degree=1/2

c. Homogeneous, degree=3 d. Homogeneous degree= 2/3

Exercise 11.1

1. There are many choices for intervals [a,b] on which f(a) and f(b) have opposite signs. The following answers are one such choice.

a. Guess two values of x for which f(x) must be opposite in signs, such as f(1)<0 and f(2)>0, so there is a root in interval [1,2]; also f(-1)<0 and f(–2)>0, so there is a root in interval [-2,-1].

2. a. c_0=1.4, c_1=1.2, c_2=1.1, c_3=1.15

3. a. $c_o = -1.8300782,\ c_1 = -1.8409252,\ c_2 = -1.8413854,\ c_3 = -1.8414048$

4. a. The iterates is converging to the actual root r_1:

$x_o = 2.5,\ x_1 = 2.41666667,\ x_2 = 2.41421569,\ x_3 = 2.41421356$

b. The iterates is converging to the actual root r_2:

$x_o = -0.5,\ x_1 = -0.41666667,\ x_2 = -0.41421569,\ x_3 = -0.414213564$

5. a. x_3=1.443 b. $x_3 = 0.000$

c. x_3=0.453 d. $x_3 = 0.000$

6. a. 0.322 b. -2.5468 c. -2.79632

7. a. Fails, since $f'(4/3) = 0$, we cannot process Newton-Raphson iterative method.

b. If $x_0 = 2$, then the first iterate is $x_1 = 0$ and it is impossible to continue with Newton-Raphson method any more.

Exercise 11.2

1. a. $T_4 = 8.75$ b. $T_4 = 1.17$

c. $T_6 = 1.11$ d. $T_6 = 3.26$

2. a. $S_6 = 18.70$ b. $S_8 = 1.11$

c. $S_6 = 1.10$ d. $S_8 = 3.19$ e. $S_4 = 0.37$

3. 3.1 5. $412\,m^2$